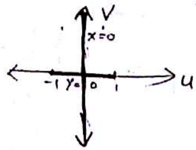
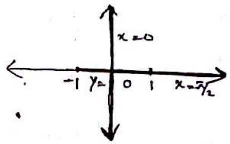
when $y=0$, the image is the line segment $-1 \le u \le 1$. ie



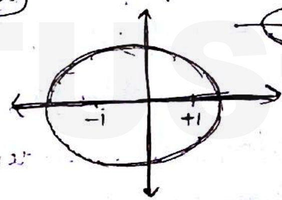
→ Now, when $x = \pi/2$, the image is $u \ge 1$. ie,



→ Now, when $y = \overset{c}{2}$, the image of the line $y=2$ is the Ellipse $u^2/\cosh^2 c + v^2/\sinh^2 c = 1$

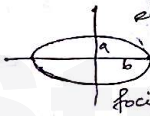
ie, $u^2/\cosh^2(2) + v^2/\sinh^2(2) = 1$

with focii $(\pm 1, 0)$



ellipse.

focii is same



MODULE - 3 ①

COMPLEX INTEGRATION

★ Some basic definitions:

(i) Line Integrals:

Complex definite integrals are called line integrals. They are of the form, $\int_C f(z)dz$ or $\oint f(z)dz$ where, C is a given curve called path of integral.

(ii) Curve:

The parametric equation $z(t) = x(t) + i\,y(t)$, $a \le t \le b$, defines a curve in the complex plane. The direction of increasing t is called positive direction on C and C is said to be an oriented curve.

(iii) Smooth Curve:

If a curve 'C' has continuous and non-zero derivatives at each point, then c is called a smooth curve. (Sharp edge - 2 curves are not smooth)

(iv) Simple curve:

A curve is simple if it doesn't intersect itself. A simple curve which is closed is called a simple closed curve.

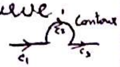
Self intersect. That is not a simple curve

$(z(a) = z(b))$, for $a \le t \le b$ ②

and for $a \le t_1, t_2 \le b$ $\;z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$

(v) Contour :

A contour is a piecewise smooth curve.



(vi) Simply Connected domain :

A domain 'D' is called simply connected, if every simple closed curve in D encloses only points of D.

eg: disc. (interior of a circle) ....

★ Properties of Line Integrals :

(i) Linearity

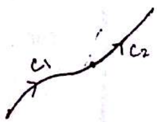
$$\int_C [K_1 f_1(z) + K_2 f_2(z)] = K_1 \int_C f_1(z)\,dz + K_2 \int_C f_2(z)\,dz$$

(ii) Sense reversal

$$\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$$

(iii) partitioning of path:

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$



⇒ Evaluation Of Line Integrals : ③

Method 1 : First Evaluation method :

Here we evaluate using the formula,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Here, $F'(x) = f(x)$

Theorem-1 :

Let $f(z)$ be analytic in a simple connected domain D. Then there exist an indefinite integral of $f(x)$ in the domain, D such that, $F'(z) = f(z)$ in D. And, for all paths in D joining two points z_0 and z_1 in D, we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

NOTE :)

Line Integral is independent on the path of integration 'C'. (depends on end points z_0 & z_1)

Examples :

? Evaluate the following.

(i) $\displaystyle\int_0^{1+i} z^2\,dz$

(ii) $\displaystyle\int_{-\pi i}^{\pi i} \cos z\,dz$

(iii) $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz$

(iv) $\displaystyle\int_{-i}^{i} dz/z$

1. $\int_0^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i}$ (A)

$= \left[\frac{1}{3}(1+i)^3 - 0\right] = \frac{1}{3}(1+3i+3i^2+i^3)$

$= \frac{1+3i-3-i}{3} = \frac{-2+2i}{3}$

$= -2/3 + 2/3\, i$.

2. $\int_{-\pi i}^{\pi i} \cos z\,dz = [\sin z]_{-\pi i}^{\pi i}$

$= \sin\pi i - \sin(-\pi i)$

$= \sin\pi i + \sin\pi i$

$= 2\sin\pi i = 2i\sinh\pi$

$= 23.097\, i$

3. $\int_{8+\pi i}^{8-3\pi i} e^{z/2}\,dz =$

$= \frac{e^{z/2}}{1/2} = \left[2e^{z/2}\right]_{8+\pi i}^{8-3\pi i}$

$= 2\left[e^{\frac{8-3\pi i}{2}} - e^{\frac{8+\pi i}{2}}\right] = 2\left[e^{4-3/2\pi i} - e^{4+\pi/2\, i}\right]$

$= 2\left[e^4 e^{-3/2\pi i} - e^4 e^{\pi/2\, i}\right] = 2e^4\left(e^{-3/2\pi i} - e^{\pi/2\, i}\right)$

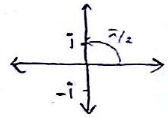
4. $\int_{-i}^{i} dz/z = [\log z]_{-i}^{i} = \log(i) - \log(-i)$ (5)

3. Continues:

$2\left[e^{4-3/2\pi i} - e^{(4-3/2\pi i)+2\pi i}\right]$ $\{e^z = e^{z+2\pi i}$

$= 2\times 0 = 0$ [Since, e^z is periodic with period $2\pi i$]

4. Continues:



i, $|i| = 1$

$\arg i = \pi/2$

$\therefore i = 1 \cdot e^{i\pi/2}$

or $\{e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 0 + i = i$

$e^{i\pi/2} = i\}$.

Substitute $i = e^{i\pi/2}$, we get

$\int_{-i}^{i} dz/z = \log(i) - \log(-i)$

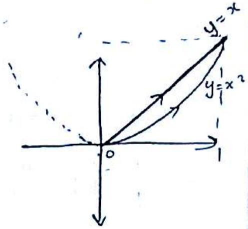
$= \log(e^{i\pi/2}) - \log(e^{-i\pi/2})$

$= i\pi/2 - -i\pi/2 = 2i\pi/2 = i\pi$

$= i\pi$

? Evaluate $\int_0^{1+i}(x^2-iy)\,dz$ along (a) $y=x$ (b) $y=x^2$. (6)

Ans. (a) along the line $y=x$:



$f(z)=x^2-i\cancel{x}y$

when $y=x$

$f(z)=x^2-ix$

$dz=dx+i\,dy$

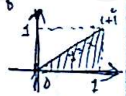
when $y=x$, $dy=dx$

$\therefore dz=dx+i\,dx=(1+i)\,dx$

$$\int_0^{1+i}(x^2-iy)\,dz=\int_0^{1+i}(x^2-ix)(1+i)\,dx$$

$$=\int^{1}(x^2+x^2i-ix-i^2x)\,dx$$

Here, x varies from 0 to 1



$$=\frac{x^3}{3}+i\frac{x^3}{3}-i\frac{x^2}{2}+\frac{x^2}{2}$$

$$=\left[\frac{x^3}{3}+i\left(\frac{x^3}{3}-\frac{x^2}{2}\right)+\frac{x^2}{2}\right]_0^1$$

$$=\frac{1}{3}+i\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{1}{2}$$

$$=\frac{1}{3}+\frac{1}{3}i-\frac{1}{2}i+\frac{1}{2}=\frac{5}{6}-\frac{1}{6}i$$

(b) along the path $y=x^2$. (7)

$f(z)=x^2-i\cancel{x}^{2}y$

when $y=x^2$, $f(z)=x^2-ix^2$

$f(z)=x^2(1-i)=x^2(1-i)$

when $y=x^2$, $dy=2x\,dx$

$dz=dx+i\,dy=dx+i\,2x\,dx$

$dz=(1+2ix)\,dx$

∴ Here, also, x varies from $0\to1$.

ie, $$\int_0^1 x^2(1-i)(1+2ix)\,dx$$

$$=(1-i)\int_0^1(x^2+2ix^3)\,dx$$

$$=(1-i)\left[\frac{x^3}{3}+2i\frac{x^4}{\cancel{4}\,2}\right]_0^1$$

$$=(1-i)\left[\frac{1}{3}+\frac{i}{2}\right]$$

$$=\frac{1}{3}-\frac{1}{3}i+\frac{i}{2}-\frac{i^2}{2}$$

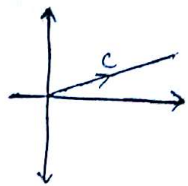
$$=\frac{5}{6}+\frac{1}{6}i$$ Both are different.

? Evaluate $\int_C z^2\,dz$ where C is the line $x=2y$ from $(0,0)$ to $(2,1)$

Here, $f(z) = z^2$ 5

$= (x+iy)^2$

$= x^2 - y^2 + i2xy$



If $x = 2y$.

$f(z) = (2y)^2 - y^2 + i\,2.2y.y$

$= 4y^2 - y^2 + i4y^2 = 3y^2 + i4y^2$

$\therefore f(z) = y^2(3+4i)$

We have $x = 2y$

$dy = dx/2 \Rightarrow dx = 2dy$

also, $dz = dx + i\,dy$

$= 2dy + i\,dy = dy(2+i)$

$dz = (2+i)dy$

(0,0) to (2,1) ⇒ x varies from $0 \to 2$ ⇒ y varies from 0 to 1. $\therefore \int_0^1$

Now $\int_C f(z)dz = \int_0^1 y^2(3+4i)\,dy\,(2+i)$

$= \int_0^1 y^2 dy\,(3+4i)(2+i)$

$= (3+4i)(2+i)\int_0^1 y^2 dy$

$= 6+3i+8i+4i^2 \left[y^3/3\right]_0^1$

$= (2+11i) \times 1/3$

$= 2/3 + 11/3\, i$

Method - 2 : Second Evaluation method : (for any fun)

This method can be applied to any continuous complex function. 9

Theorem:

Let 'C' be a piecewise smooth path represented by $z = z(t)$, $a \le t \le b$. Let $f(z)$ be a continuous function on C, then $\int_C f(z)dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt$, where $\dot{z}(t) = dz/dt$

Algorithm.

Step 1 : Represent the path C in the form $z(t)$, $a \le t \le b$

Step - 2 : Calculate the derivative $\dot{z}(t)$

Step - 3 : Substitute $z(t)$ for every z in $f(z)$. (Hence, $x(t)$ for x and $y(t)$ for y)

Step - 4 : Integrate $f[z(t)]\,\dot{z}(t)$ with respect to 't' from a to b. $z(t) = x(t) + i\,y(t)$

? Evaluate $\int_C dz/z$, where, C is unit circle in anticlockwise direction.

Step-1:
Ans. Here, C is the unit circle in counterclockwise direction. Its parametric form is $z(t) = e^{it}$, $0 \le t \le 2\pi$

Step-2:
Then $\dot{z}(t) = ie^{it}$

we have, $f(z) = 1/z$

Now, $f[z(t)] = 1/z(t) = 1/e^{it} = e^{-it}$

$\therefore \int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$

$\therefore \int \frac{dz}{z} = \int_0^{2\pi} e^{-it} \cdot ie^{it}\, dt$

$= i\int_0^{2\pi} 1\, dt$

$= i(t)_0^{2\pi}$

$\int_C \frac{dz}{z} = 2\pi i$

Result:

$$\boxed{\int_{|z|=1} \frac{dz}{z} = 2\pi i}$$

ie, $\int \frac{dz}{z}$ over unit circle is $2\pi i$

$|z| = 1$, unit circle

? Evaluate $\int_C e^z dz$, where C is the shortest path from πi to $2\pi i$.

$$\int_C e^z dz = \int_{\pi i}^{2\pi i} e^z dz = [e^z]_{\pi i}^{2\pi i}$$

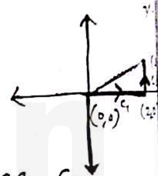
$= e^{2\pi i} - e^{\pi i}$

$= e^{\pi i} \cdot e^{\pi i} - e^{\pi i}$

$= e^{\pi i}(e^{\pi i} - 1)$

? Evaluate $\int_C z^2 dz$, where C is given by line along the real axis from (0,0) to (2,0) and then vertically to (2,1)

Ans.



(take the path from (0,0) to (2,0) as C_1 and the path from (2,0) to (2,1) as C_2

Along the path C_1 $(y=0)$:

Here, $y = 0$, ie, $z^2 = x^2 - y^2 + 2ixy$

$z^2 = x^2$

when $y = 0$, $dy = 0$

$\therefore dz = dx + idy$

$dz = dx$ (here, x varies from 0 to 2)

$\therefore \int_{C_1} z^2 dz = \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} = \frac{8}{3}$

12

Along C_2 : $(x = 2)$

$\therefore z^2 = x^2 - y^2 + 2ixy$

$= 4 - y^2 + 4iy$

$\therefore z^2 = 4 - y^2 + 4iy$

we have, $x = 2$

$dx = 0$

Now, $dz = dx + idy \Rightarrow dz = idy$. { y varies from $0 \to 1$.

$$\therefore \int_{C_2} z^2 dz = \int_0^1 (4 - y^2 + 4iy)\, i\, dy$$

$$= i\left[4y - \frac{y^3}{3} + 4i\frac{y^2}{2}\right]_0^1$$

$$= i\left[4 - \frac{1}{3} + \cancel{4}^{2} i/\cancel{2}\right] = 4i - \frac{1}{3}i + 2i^2$$

$$= -2 + 4i - \frac{1}{3}i = -2 + \frac{11}{3}i$$

$$\therefore \int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz$$

$$= \frac{8}{3} - 2 + \frac{11}{3}i$$

$$\int_C z^2 dz = \frac{2}{3} + \frac{11}{3}i$$

z^2 fn is analytic. ie, it independent on path. $\int z^2$ is always $\frac{2}{3} + \frac{11}{3}i$

? Evaluate $\int_C \bar{z}\, dz$, where C is parametrized by $z(t) = 3t + it^2$, $-1 \le t \le 4$

13

$\dot{z}(t) = 3 + i \times 2t$

$f(z) = \bar{z}$

$f[z(t)] = \overline{z(t)} = \overline{3t + it^2} = 3t - it^2$

$$\therefore \int_C \bar{z}\, dz = \int_{-1}^{4} (3t - it^2)(3 + i2t)\, dt$$

$$= \int_{-1}^{4} (9t + 6it^2 - 3it^2 - i^2 2t^3)\, dt$$

$$= \left[9\frac{t^2}{2} + \cancel{6}^{2} i \frac{t^3}{\cancel{3}} - \cancel{3} i \frac{t^3}{\cancel{3}} + \cancel{2}\frac{t^4}{\cancel{4}_2}\right]_{-1}^{4}$$

$$= \left(9 \times \frac{16}{2} + 2i \cdot 64 - i\, 64 + \frac{256}{2}\right) - \left(9 \times \frac{1}{2} + 2i \times -1 - i \times -1 + \frac{1}{2}\right)$$

$$= (9 \times 8 + i\,64 + 128 - 4.5 + i - 1/2)$$

$$= (72 + 64i + 128 - 5 + i)$$

$$= 195 + 65i$$

$f(z) = \bar{z}$

$f(z(t)) = \overline{z(t)} = 3t - it^2$

$\int f(z(t))\, \dot{z}(t)$

$-4.5 - 0.5$

$128 + 72 = 200$

Cauchy's Integral theorem:

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path 'C' in D, $\oint_C f(z)\, dz = 0$

for Example,

for any closed path C, (i) $\oint_C e^z\, dz = 0$ (e^z is analytic) (14)

(ii) $\oint_C \cos z\, dz = 0$

(iii) $\oint_C z^n\, dz = 0$

Theorem:

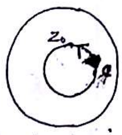
If $f(z)$ is analytic in a simply connected domain D, then the $\int f(z)$ is independent ~~of the~~ of path in D.

★ Cauchy's Integral formula:

Theorem:

Let $f(z)$ be an analytic function in a simply connected domain D, then for any point z_0 in D and any simple closed path 'C' in D that encloses z_0,

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0).$$



The integration being taken counterclockwise.

Alternatively, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\, dz$

Example;

? Evaluate $\int_C \frac{e^z}{z-2}\, dz$, where, C is a simple closed curve enclosing $z_0 = $

(15)

Ans. Now,

By Cauchy's Integral formula,

$$\oint \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$$

Now, here $f(z) = e^z$ (which is analytic)

$f(2) = e^2$

$\therefore \oint_C e^z/_{z-2}\, dz = 2\pi i \times e^2 = 46.4268 i$

? Evaluate $\oint_C \frac{z^3 - 6}{2z - i}\, dz$

Ans. $\oint_C z^3 - 6/_{2z-i}\, dz = \oint_C \frac{z^3/2 - 6/2}{z - i/2}\, dz$ (divide 2)

$= \oint \frac{z^3/2 - 3}{z - i/2}\, dz$

$= 2\pi i\, f(i/2)$ $\{z_0 = i/2\}$

$= 2\pi i \left(\frac{(i/2)^3}{2} - 3\right)$ $f(z) = z^3/2 - 3$

$= 2\pi i \left(\frac{i^3}{16} - 3\right)$

$= 2\pi i \times \left(\frac{-i}{16} - 3\right) = \frac{-2\pi i^2}{16} - 2\pi i \cdot 3$

$= \pi/8 - 6\pi i$

? Evaluate $\oint_C \frac{\cos \pi z}{z-1}\, dz$, where c is the circle $|z| = \frac{3}{4}$

(16)

Ans: Here $z_0 = 1$, $\cos \pi z = f(z)$

Also, $z_0 = 1$ lies inside c. Hence, by Cauchy's integrals formula

ie, $\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0) = 2\pi i f(1)$

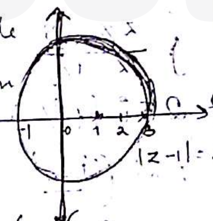
$= 2\pi i \times (\cos \pi)$

$= -2\pi i$

$\therefore \oint \frac{\cos \pi z}{z-1}\, dz = -2\pi i$ //

? Evaluate $\oint_C \frac{z+2}{z-2}\, dz$, where c is the circle $|z-1| = 2$

Ans: Here, $z_0 = 2$ is lies inside the circle. Hence, we can apply Cauchy's integral formula



the point $z_0 = 2$ should be within the circle. Otherwise, we can't apply the Cauchy integral formula

$\therefore \oint \frac{z+2}{z-2}\, dz = 2\pi i\, f(2)$

$= 2\pi i \times 4 = 8\pi i$ //

(17)

Derivatives of Analytic functions:

Theorem:

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders, which are then also analytic functions in D.

The values of these derivatives at a point z_0 in D are given by the formulae

(i) $f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz$

(ii) $f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\, dz$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

General form of Cauchy's integral formula

? Evaluate $\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\, dz$ where c is the circle $|z| = 1$

Ans $\oint \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i}{2!}$

$= \frac{2\pi i}{2!} \times f''(\pi/6)$

Here $f(z) = \sin^2 z$

$f(z_0) = f(\pi/6) = \sin^2 \pi/6$

$f'(z_0) = 2 \sin \pi/6 \cos \pi/6 = \sin 2\pi/6 = \sin \pi/3$

$f''(z_0) = 2 \cos 2\pi/6$

$= 2 \cos \pi/3 = 2 \times 1/2 = 1$

$\therefore \oint \frac{\sin^2 z}{(z-\pi/6)^3} = \pi i$

? Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where, C is the circle $|z| = 2$.

Ans. $f(z) = e^{2z}$

$f'(z) = e^{2z} \times 2$

$f''(z) = 2e^{2z} \times 2 = 4e^{2z}$, $f'''(z) = 8e^{2z}$

MODULE-3 : Continuous ...... (19)

? Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along the line $y = x/2$.

Ans. $f(z) = (\bar{z})^2$

$(x - iy)^2$
$x^2 - y^2 - 2ixy$
$(2y)^2 - y^2 - 2i(2y)y$
$4y^2$
$3y^2 - 4y^2 i$
$y^2(3-4i)$

Along the line $y = x/2$, $x = 2y$.

$\therefore z = x + iy = 2y + iy = y(2+i) = f(z)$

But, we have $f(z) = (\bar{z})^2$

$\therefore f(z) = \left(\overline{(2+i)y}\right)^2 = (2-i)^2 \cdot y^2$

$dz = dx + i\,dy = 2dy + i\,dy = (2+i)\,dy$

$z = (2+i)(y) \Rightarrow dz = (2+i)\,dy$

When $z = 0$,

$z = (2+i)y \Rightarrow y = z/(2+i)$

when $z = 0$, $y = 0$

when $z = 2+i$, $y = 1$.

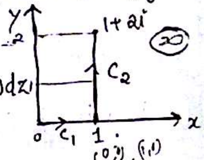
$$\therefore \int_0^{2+i} (\bar{z})^2 dz = \int_0^1 (2-i)^2 \cdot y^2 (2+i)\, dy$$

$$= \int_0^1 (2-i)\underbrace{(2-i)(2+i)}_{2^2 - i^2} y^2 \, dy$$

$$= (2-i)\, 5 \left[\frac{y^3}{3}\right]_0^1 = (2-i)\, 5/3$$

$$= 5/3 \times 2 - 5/3\, i = 10/3 - 5/3\, i$$

? Evaluate $\int_C Re(z)\, dz$, along the path C, consisting of paths C_1 and C_2 as shown in the figure (20)



Ans. $\int_C Re(z)\,dz = \int_{C_1} Re(z)\,dz + \int_{C_2} Re(z)\,dz$

$(0,0), (1,0)$

$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

Along C_1:

$z(t) = t$, $0 \le t \le 1$.

Then $\dot{z}(t) = 1$

$f[z(t)] = Re(z(t))$

$= Re[x(t) + i y(t)]$

$= x(t)$

$$\therefore \int_{C_1} Re(z)\, dz = \int_0^1 f(z(t)) \cdot \dot{z}(t)\, dt$$

$$= \int_0^1 x(t) \cdot 1 \cdot dt$$

$$= \int t\, dt = \left[t^2/2\right]_0^1 = 1/2$$

$z(t) = t$
$x(t) + i y(t) = t + i0$
ie, $x(t) = t$

Along C_2:

y varies from $0 \to 2$

$z(t) = 1 + it$, $0 \le t \le 2$

$\dot{z}(t) = i$

$f[z(t)] = Re(1 + it) = 1$

$(1,0), (1,2)$

$\frac{x-1}{1-1} = \frac{y-0}{2-0}$

$2(x-1) = 0$

$2x - 2 = 0$

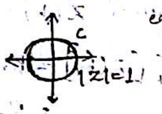
Therefore, $\int_{C_2} Re(z)\,dz = \int_0^2 1 \cdot i\,dt$ (21)

$= i\,[t]_0^2 = 2i$

$\therefore \int_C Re(z)\,dz = 1/2 + 2i$

? Evaluate $f(z) = e^{-z^2}$ along the unit circle in the counter clockwise direction.

Ans. we have $f(z) = e^{-z^2}$



eqn of unit circle, $|z|=1$.

$f(z) = 1/e^{z^2}$

The given fn is analytic in the entire complex plane

$\int_C f(z)\,dz = \int_{|z|=1} e^{-z^2}\,dz$

$\int_C f(z)\,dz = 0$, Since by Cauchy's Integral formula

Here, $f(z) = e^{-z^2}$ is analytic in the entire complex plane.

The unit circle $|z| = 1$ is a simple closed path in a simple complex plane, therefore by Cauchy's integral theorem,

$\oint_C f(z)\,dz = 0$

ie, $\int_{|z|=1} e^{-z^2}\,dz = 0$ (22)

~~? Evaluate $\oint_C \frac{z+2}{z-2}\,dz$, where C is $|z-1| = 2$~~

? Evaluate $\oint_C \frac{\sin z}{z^4}\,dz$ where C is the Circle $|z| = 1$ (unit circle).

Ans. $f(z) = \sin z$

$z_0 = 0$ ie, $\oint \frac{\sin z}{(z-0)^4}\,dz$

We use $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(z_0)$

$\therefore \oint \frac{\sin z}{(z-0)^4}\,dz = \frac{2\pi i}{3!} f'''(0)$ {n = 3 (4-1)

$f(z) = \sin z$, $f'(z) = \cos z$, $f''(z) = -\sin z$

$f'''(z) = -\cos z$

$\therefore f'''(0) = -\cos 0 = -1$

$\therefore \oint \frac{\sin z}{(z-0)^4}\,dz = \frac{\cancel{2}\pi i}{\cancel{6}\,3} \times -1$

$= \frac{-\pi i}{3}$

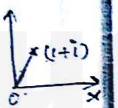
★ ML-INEQUALITY: (23)

This inequality help us to find an upper bound for the absolute value of a complex line integral and is given by,

$$\left|\int_C f(z)\,dz\right| \le ML$$

, where M is a constant such that $|f(z)| \le M$ everywhere on C. And L is the length of the path/arc 'C'.

? Find an upper bound for the absolute value of integral $\int_C z^2\,dz$, where C is the straight line segment from '0' to '1+i'.



Ans. $f(z) = z^2$

$|f(z)| = |z^2| \le |(1+i)^2|$

(max |·| is for 1+i (max distance))

$|z^2| \le 2$

ie, $M = 2$

$(1+i)^2$
$1-1+2i$
$=2i$
$|2i| = 2.$

Now, L is the length of C.

ie, distance to 1+i.

ie, $|1+i| = \sqrt{2}$

∴ $L = \sqrt{2}$ //

$(0,0)\to(1,1)$
$\sqrt{(1-0)^2+(1-0)^2} = \sqrt{1+1}$

...refore, by ML inequality,

$\left|\int_C z^2\,dz\right| \le ML = 2\sqrt{2}$ // 2.8284 // (24)

? Integrate $z^2/(z^2-1)$ by Cauchy's formula counterclockwise around the circle,

(a) $|z+1| = 1$ (b) $|z-1-i| = \pi/2$ (c) $|z+i| = 1.4$
(d) $|z+5-5i| = 7$.

Ans. we have, the integrand is analytic except at $z=1$ and $z=-1$

Here C is (a) $|z+1| = 1$

$|z-(-1)| = 1$

radius 1, centre $= -1$

$|z-z_0| = r$ at centre z_0 radius 'r'

here, $z=1$ lies outside C and

$z=-1$ lies inside C.

Since, $z=-1$ lies inside C,

∴ $f(z) = z^2/(z-1)$

ie, $\frac{z^2}{z^2-1} = \frac{z^2}{(z+1)(z-1)}$
$\Rightarrow \frac{z^2/(z-1)}{z+1}$

choose $f(z) = z^2/(z-1)$

ie, $f(z)$ is analytic inside and on C

$z^2/(z-1)$ is not analytic at $z=1$. Since $z=1$ lies outside, ∴ it is analytic inside & on C.

∴ by Cauchy's integral formula,

$$\int_C \frac{z^2}{z^2-1}\,dz = \int \frac{z^2/(z-1)}{z+1}\,dz$$

$= 2\pi i f(z_0)$

$= 2\pi i\, f(-1) = 2$ (25)

$= 2\pi i \times -1/2$

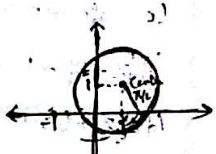
$= -\pi i$

$\left\{ \begin{array}{l} \because f(z) = z^2/z-1 \\ f(-1) = 1/-2 \end{array} \right.$

(b) $|z-1-i| = \pi/2$

$|z-(1+i)| = \pi/2$

Centre $=(1+i)$



ie, $z = 1$ lies inside the circle

and $z=-1$ lies outside the circle

$\left\{ \begin{array}{l} (1,1) \\ (-1,0) \\ \sqrt{2^2+1^2} = \sqrt{5} > \pi/2 \\ \therefore -1 \text{ is outside} \end{array} \right.$

Since, $z = 1$ lies inside the the circle, so, $z_0 = 1$

$\therefore f(z) = z^2/z+1$

$\therefore \; z^2/z^2-1 = \dfrac{z^2/z+1}{z-1}$ $\quad z = z_0$

$\therefore$ By cauchy's integral formula,

$\therefore \displaystyle\int_C \frac{z^2}{z^2-1}\, dz = \int \frac{z^2/z+1}{z-1}\, dz = 2\pi i\, f(z_0)$

$= 2\pi i \times 1/2$

$= \pi i$

$f(z) = z^2/z+1$

$f(1) = 1^2/2 = 1/2$

(c) $|z+i| = 1.4$ (26)

$|z-(-i)| = 1.4 \Rightarrow$ centre $(-i)$

radius $= 1.4$

here, $r = 1.4$

Now, consider the centre $(0,-1)$

and the point $(1,0)$

by distance formula, $\sqrt{(1-0)^2+(0-1)^2}$

$= \sqrt{2} = 1.41$

$\therefore$ actual value of $\sqrt{2}$ is $>$ than 1.4.

$\therefore z = 1$ lies outside the circle.

$(0,-1)$, $(-1,0)$ $= \sqrt{(-1-0)^2+(0-1)^2} = \sqrt{2}$

$\therefore z=-1$ also lies outside the circle.

(therefore, we can't use cauchy's integral formula)

Hence, by cauchy's integral theorem,

take $f(z) = z^2/z^2-1$ which is analytic inside and on C. {it is not analytic at $z = 1, -1$}

$\therefore$ it is analytic inside and on C.

Hence by cauchy's integral theorem,

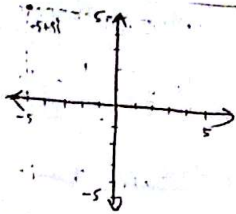
$\displaystyle\int z^2/z^2-1\, dz = 0$

(4) $|z+5-5i| = 7$

$|z-(-5+5i)| = 7$

(27)

centre, $-5+5i$, radius $= 7$



$\therefore z=-1$ lies inside C

and $z=1$ lies outside C.

$(-5,5)$, $(-1,0)$

$\sqrt{(-1+5)^2+(0-5)^2}$

$= \sqrt{16+25} = \sqrt{41}$

$z=-1$ lies inside C.

for $z^2/z-1$, which is analytic inside C since, $z=1$ lies outside the circle only at point $z=1$ it is not analytic

$\therefore f(z) = \dfrac{z^2}{z^2-1}$

$z-z_0$

$z-(-1)$

$z- -1$

$f(z) = \dfrac{z^2/z-1}{z+1}$

$(-5,5) \to (1,0)$

$\sqrt{6^2+5^2} = \sqrt{36+25}$

$= \sqrt{61}$

$\therefore z=1$ lies ~~inside~~ outside C.

Then by Cauchy's integral formula;

$\displaystyle\int z^2/z^2-1 \, dz = \int \frac{z^2/z-1}{z+1} dz$

$f(z) = z^2/z-1$

$f(-1) = 1/-2$

$= 2\pi i \, f(z_0) = 2\pi i \times \frac{1}{-2}$

$= -\pi i$ //

HW Evaluate $\displaystyle\int_C z/z^2-1 \, dz$ where C is

(a) $|z| = 1/2$

(b) $|z-1| = 1/2$

(c) $|z+1| = 1/2$

(d) $|z| = 2$

Ans. (a) The integral is analytic except $z=1$, $z=-1$

(28)

$\displaystyle\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$

(a) Here, the circle $|z| = 1/2$

Hence, $z=1$ & $z=-1$ lies outside the C,

take $f(z) = z/z^2-1$, which is analytic inside and on C.

Hence, by Cauchy's integral theorem;

$\displaystyle\int_C z/z^2-1 \, dz = 0.$

(b) Here, the circle $|z-1| = 1/2$.

Hence $z=1$, lies inside the circle and $z=-1$ lies outside the circle.

choose $f(z) = z/z+1$

$f(z)$ is analytic inside and on C.

$\therefore$ By Cauchy's integral formula,

$\frac{1}{1+1}$

$\displaystyle\int_C z/z^2-1 = \int_C \frac{z/z+1}{z-1} dz$

$= 2\pi i f(z_0) = 2\pi i f(1)$

$= 2\pi i \times 1/2 = \pi i$ //

(c) Here, the circle $|z+1| = 1/2$

Hence, $z=1$ lies outside the circle and $z=-1$ lies inside the circle.

(29)

Choose $z=-1$ lies inside the circle.

choose, $f(z) = z^2/z-1$ $\quad \frac{z^2}{z^2-1} = \frac{z^2}{(z+1)(z-1)}$

$f(z)$ is analytic inside and on C, $\frac{z^2}{z-z_0} = \frac{z^2/z-1}{z+1}$

∴ By cauchy's integral formula,

$$\int_c \frac{z^2}{z^2-1} dz = \int_c \frac{z^2/(z-1)}{z+1} dz$$

$$= 2\pi i f(z_0) - 2\pi i f(-1)$$

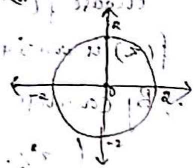
$$= 2\pi i \times (-1)^2/_{-1-1} = \frac{2\pi i}{-2} = +\pi i //$$

(d) Here, the circle $|z|=2$ $\quad |z-z_0|=r$, $|z-0|=2$

Hence both $z=1$ & $z=-1$ lies inside the circle, $|z|=2$.

$\cancel{\frac{z}{z^2-1} = \frac{z^2}{(z+1)(z-1)}}$

$\cancel{\frac{A}{z+1} + \frac{B}{z-1} = \frac{z}{z+1}}$



Take $f(z) = z/z^2-1$

Using partial fractions,

$$\frac{z}{z^2-1} = \frac{z}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$z = A(z+1) + B(z-1)$, $z = z(A+B) + A-B$, $A+B=1$, $A-B=0$, $A=B$, $2A=1$, $A=1/2$, $B=1/2$

$1 = z(A+B) + (A-B)$, $A+B=1$, $B+B=1$, $2B=1$, $A-B=0$

From this, $A = +1/2$, $B = 1/2$

(30)

$$\therefore \frac{z}{z^2-1} = \frac{z/2}{z+1} + \frac{z/2}{z-1}$$

$$\int_c \frac{z}{z^2-1} dz = \int_c \frac{z/2}{z-1} dz + \int_c \frac{z/2}{z+1} dz$$

$$= 2\pi i f(1) + 2\pi i f(-1)$$

$$= 2\pi i f\ 1/2 + 2\pi i \times -1/2$$

$$= \pi i - \pi i = 0 //$$

Q? Evaluate (i) $\int \frac{z^2+1}{z^2-1} dz$ over the circle $|z-1|=1$

(ii) $\int_c \frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)(z-2)} dz$, where C is $|z|=3$

(iii) $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z|=2$

(iv) $\int_c \frac{z^2}{(z-1)^2 (z+2)} dz$, where $|z-2| = 2$.

$\frac{z^2/z+2}{(z-1)^2} = \frac{2\pi i}{1!} f'(z_0)$

(v) $\int_c \frac{\sin^2 z}{(z-\pi/6)^3} dz$, where C is $|z|=1$

(vi) $\int \frac{e^z}{(z+2)(z+1)^2} dz$ $\quad |z|=3$

(7) $\int_c \frac{z}{(z-1)(z-2)^2} dz$, C is $|z-2|=1/2$

(i) $2\pi i$
(ii) $4\pi i$
(iv) $\frac{10\pi i}{9}$
(vi) $2\pi i$
(vii) $-2\pi i$

★ TAYLOR & MACLAURIAN SERIES:

(31)

① The taylor series of a complex function $f(z)$ is, $f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n$, where $a_n = \frac{1}{n!} f^{(n)}(z_0)$

By cauchy's integral formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$$

$$\therefore, a_n = \frac{1}{n!} \times \frac{n!}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^* \Rightarrow$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$$

A maclaurian series is a taylor series with $z_0 = 0$. Hence, the maclaurian series of $f(z)$ is,

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{n!} f^{(n)}(0)$$

The reminder of taylor series after the term $a_n (z-z_0)^n$ is,

$$\cancel{f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n = a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n + [\dots]}$$

denoted by, $R_n(z)$ is given by,

(32)

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n = a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n + \underbrace{[\dots\dots]}_{R_n(z)}$$

②

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$

Thus,

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \dots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z)$$

The above formula is called Taylor's formula with remainder.

★ TAYLORS THEOREM:

Let $f(z)$ be analytic in a domain D and z_0 be any point in D, then there exist precisely one taylor series $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ with centre at z_0 that represents $f(z)$. This representation is valid in the largest open disc with

Centre, z_0 in which $f(z)$ is analytic. (33)

The coefficients satisfies the inequality $|a_n| \le \frac{M}{r^n}$ where, M is the max. of $f(z)$ on the circle $|z-z_0| = r$, in D whose interior is also in D.

Some special Taylor Seriases:

(i) Geometric Series:

The Maclaurian series expansion of $f(z) = 1/_{1-z}$, is called the geometric series and is given by,

$$1/_{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3 \cdots\cdots$$

(ii) Exponential function:

The maclaurian series expansion of

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots$$

(iii) Trigonometric function:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots$$

(iv) Hyperbolic functions: (34)

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

(v) Logarithmic functions:

$$\ln(1+z) = z - z^2/_2 + z^3/_3 - \cdots\cdots$$

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\cdots\right)$$

$$\ln(1-z) = \ln(1+(-z)) = -z - \frac{z^2}{2} - \frac{z^3}{3}$$

$$-\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\cdots$$

$$\left\{ \ln(1+z) - \ln(1-z) = \ln\left(\frac{1+z}{1-z}\right) \right\}$$

? Write $f(x) = \sin x$ as a Taylor series about

Ans. $\sin z = f(z)$; $f(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$

$f'(z) = \cos z$; $f'(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$

$f''(z) = -\sin z$, ; $f''(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$

$f'''(z) = -\cos z$; $f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$

By Taylor's theorem, f.

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$ (35)

$$= f(\pi/4) + (z-\pi/4) f'(\pi/4) + \frac{(z-\pi/4)^2}{2} f''(\pi/4) + \cdots$$

$$f(z) = 1/\sqrt{2} + (z-\pi/4) \times 1/\sqrt{2} - \frac{(z-\pi/4)^2}{2} \times 1/\sqrt{2} + \cdots$$

$$f(z) = 1/\sqrt{2}\left[1 + (z-\pi/4) - (z-\pi/4)^2 + \cdots\right]$$

? Find the maclaurian series of $f(z) = \frac{1}{1+z^2}$.

Ans Maclaurian series $\Rightarrow$ (at $z=0$).

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum (-z^2)^n = \sum (-1)^n z^{2n}$$

we have $\left\{ 1/1-z = 1 + z + z^2 + z^3 + \cdots \right\}$

$$\therefore 1/1-(-z^2) = 1 + -z^2 + (-z^2)^2 + (-z^2)^3$$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

$$\left\{ \text{Since, } \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots \quad \sum_{n=0}^{\infty} z^{2n} = 1 + z^2 + z^4 + z^6 + \cdots \right\}$$

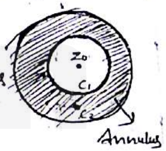
Module 4 : (36)

★ LAURENT SERIES :

We use laurent series to develop a fun^n $f(z)$ in powers of $z-z_0$, when $f(z)$ is singular at z_0. Laurent series consists of both +ve and -ve powers of $z-z_0$ and the series converges in an annulus with centre at z_0.

⇒ Laurent's theorem :

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with centre at z_0 and the annulus b/w them. Then $f(z)$ can be represented by the laurents series



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots + \underbrace{\left\{ b_1/(z-z_0) + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots \right\}}_{\text{principal part}}$$

where, $a_n = 1/2\pi i \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$

and $b_n = 1/2\pi i \oint_C (z^*-z_0)^{n+1} f(z^*) dz^*$

Taken counter clockwise around any simple closed path 'c' that lies in the annulus and encircles the inner circles.

NOTE:

(i) A fun^n $f(z)$ can have several Laurents series expansion with the same centre z_0 and valid in several concentric annuli.

Taylor series expansion about a point is unique But the Laurents series expansion not unique at a point

(ii) The series of the negative powers of Laurents series is called principal part of the singularity of $f(z)$ at z_0.

(iii) the coefficient of $1/(z-z_0)$ in the Laurent's series expansion is called the residue of $f(z)$ at z_0.

? Find the Laurent's series of $z^{-5}\sin z$ with centre 0.

Ans- Maclaurian series for $\sin z = z - z^3/3! + z^5/5! + \cdots$

ie, $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

$$z^{-5}\sin z = z^{-5}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$z^{-5} z^{2n+1} = z^{2n-4}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n-4}}{(2n+1)!}$$

This exp^n contains both -ve & +ve powers of z

ie, $\underbrace{\frac{1}{z^4} - \frac{1}{6z^2}}_{\text{principal part (-ve powers of z)}} + \frac{1}{5!} - \frac{z^2}{7!} + \frac{z^4}{9!} - \frac{z^6}{11!} + \cdots$

is a Laurents series exp^n.

? Find the Laurent's series of $z^2 e^{1/z}$ with centre 0.

Ans. M. series for $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$

ie, for $e^{1/z} = \sum_{n=0}^{\infty}\frac{(1/z)^n}{n!}$

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{z^n n!}$$

But, our given fun^n is, $z^2 e^{1/z}$

ie, $z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty}\frac{1}{z^n n!}$

$$= \sum_{n=0}^{\infty}\frac{z^2}{z^n n!} = \sum_{n=0}^{\infty}\frac{z^{2-n}}{n!}$$

ie, $= \frac{z^2}{0!} + \frac{z^1}{1!} + \frac{1}{2!} + \frac{z^{-1}}{3!} + \frac{z^{-2}}{4!} + \cdots$

$$= z^2 + z + 1/2! + \underbrace{\frac{1}{3!\,z} + \frac{1}{24z^2} + \cdots}_{\text{Principal part}}$$

Here, the residue is $1/6$ (Coefficient of $1/z$).

? Expand $f(z) = \frac{z}{(z+1)(z+2)}$ in Laurent's series about (a) $z=-2$. (89)

Ans. $f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$

are the points at which the z is not analytic

$$f(z) = \frac{-1}{z+1} + \frac{2}{z+2}$$

$$= \frac{2}{z+2} - \frac{1}{z+1}$$

$$= \frac{2}{z+2} - \frac{1}{z+2-1}$$

$$= \frac{2}{z+2} + \frac{1}{1-(z+2)}$$

$$f(z) = \underbrace{\frac{2}{z+2}}_{\text{Principal part}} + 1 + (z+2) + (z+2)^2 + \dots\dots$$

$z = A(z+2) + B(z+1)$

$A+B=1$

$2A+B=0$

$A=-1$

$B=2$

$(1+x)^{-1} = 1-x+x^2-x^3+\dots$

$(1-x)^{-1} = 1+x+x^2 \dots$

$a-b = -(b-a)$

$z+2-(1-(z+2))$

Here, residue is 2.

? Find the Laurent's Series of $1/z^3 - z^4$ with centre 0 or about $z=0$

Ans. $f(z) = \frac{1}{z^3(1-z)}$

multiplying by $1/z^3$ i.e.

$$f(z) = \frac{1}{z^3} \times \frac{1}{1-z} \qquad (90)$$

$$= \frac{1}{z^3}(1+z+z^2+z^3 \dots\dots)$$

(ii) $$f(z) = \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}}_{\text{Principal part}} + 1 + z + z^2 + \dots \qquad |z|<1$$

residue is 1. (Coefficient of $1/z$).