Sorting Algorithms

Sorting

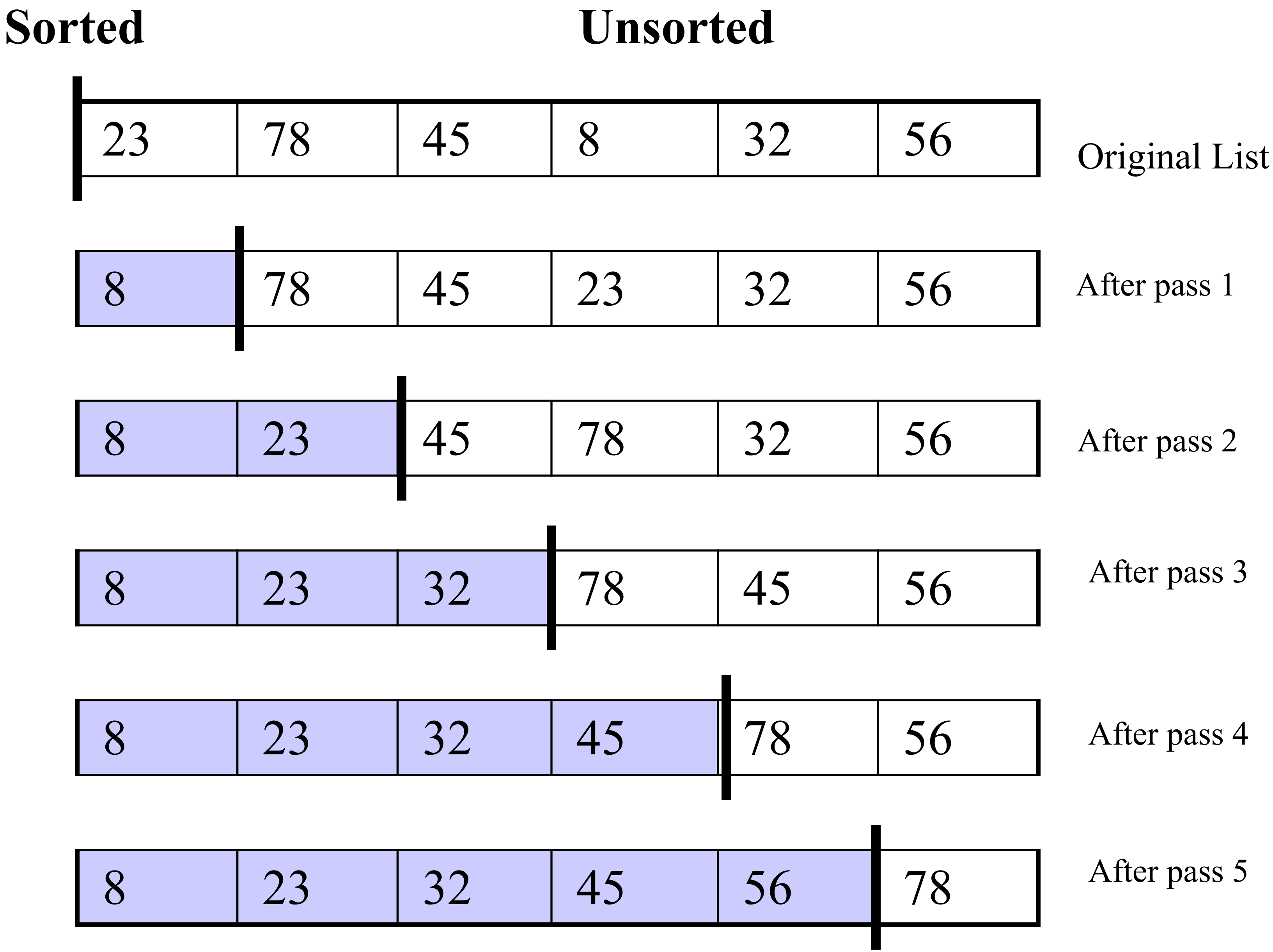
- ***Sorting*** is a process that organizes a collection of data into either ascending or descending order.
- An ***internal sort*** requires that the collection of data fit entirely in the computer's main memory.
- We can use an ***external sort*** when the collection of data cannot fit in the computer's main memory all at once but must reside in secondary storage such as on a disk.

Sorting Algorithms

- There are many sorting algorithms, such as:
  - Selection Sort
  - Insertion Sort
  - Bubble Sort
  - Merge Sort
  - Quick Sort
  - Heap Sort
- The first three are the foundations for faster and more efficient algorithms.

Selection Sort

- The list is divided into two sublists, *sorted* and *unsorted*, which are divided by an imaginary wall.
- **We find the smallest element from the unsorted sublist and swap it with the element at the beginning of the unsorted data.**
- After each selection and swapping, the imaginary wall between the two sublists move one element ahead, increasing the number of sorted elements and decreasing the number of unsorted ones.
- Each time we move one element from the unsorted sublist to the sorted sublist, we say that we have completed a sort pass.
- A list of *n* elements requires *n-1* passes to completely rearrange the data.



Sorted
Unsorted
23 78 45 8 32 56
Original List
8 78 45 23 32 56
After pass 1
8 23 45 78 32 56
After pass 2
8 23 32 78 45 56
After pass 3
8 23 32 45 78 56
After pass 4
8 23 32 45 56 78
After pass 5

Selection Sort (cont.)

```
void selectionSort( Item a[], int n)
 {
  for (int i = 0; i < n-1; i++)
  {
    int min = i;
    for (int j = i+1; j < n; j++)
        {
        if (a[j] < a[min])
        min = j;
        }
    swap(a[i], a[min]);
  }
}
```

Selection Sort -- Analysis

- In general, we compare keys and move items (or exchange items) in a sorting algorithm (which uses key comparisons).

  So, to analyze a sorting algorithm we should count the number of key comparisons and the number of moves.

  - Ignoring other operations does not affect our final result.

- In selectionSort function, the outer for loop executes n-1 times.
- We invoke swap function once at each iteration.
  - Total Swaps: n-1
  - Total Moves: 3*(n-1) (Each swap has three moves)

Selection Sort – Analysis (cont.)

- The inner for loop executes the size of the unsorted part minus 1 (from 1 to n-1), and in each iteration we make one key comparison.
  - # of key comparisons = 1+2+...+n-1 = n*(n-1)/2
  - **So, Selection sort is O(n^2)**
- The best case, the worst case, and the average case of the selection sort algorithm are same. all of them are **O(n^2)**
  - This means that the behavior of the selection sort algorithm does not depend on the initial organization of data.
  - Since O(n^2) grows so rapidly, the selection sort algorithm is appropriate only for small n.
  - Although the selection sort algorithm requires O(n^2) key comparisons, it only requires O(n) moves.
  - A selection sort could be a good choice if data moves are costly but key comparisons are not costly (short keys, long records).

Comparison of *N*, *logN* and *N*^{2}

N	O(LogN)	O(N^2)
16	4	256
64	6	4K
256	8	64K
1,024	10	1M
16,384	14	256M
131,072	17	16G
262,144	18	6.87E+10
524,288	19	2.74E+11
1,048,576	20	1.09E+12
1,073,741,824	30	1.15E+18

Insertion Sort

- Insertion sort is a simple sorting algorithm that is appropriate for small inputs.
  - Most common sorting technique used by card players.
- The list is divided into two parts: sorted and unsorted.
- In each pass, the first element of the unsorted part is picked up, transferred to the sorted sublist, and inserted at the appropriate place.
- A list of *n* elements will take at most *n-1* passes to sort the data.

Sorted **Unsorted**

23	78	45	8	32	56	Original List
23	78	45	8	32	56	After pass 1
23	45	78	8	32	56	After pass 2
8	23	45	78	32	56	After pass 3
8	23	32	45	78	56	After pass 4
8	23	32	45	56	78	After pass 5

Insertion Sort Algorithm

```
void insertionSort(Item a[], int n)
{
   for (int i = 1; i < n; i++)
   {
      Item tmp = a[i];

      for (int j=i; j>0 && a[j-1]> tmp; j--)
         {a[j] = a[j-1];}

      a[j] = tmp;
   }
}
```

Can do search from 1st element in inner for loop ..but then best case efficiency will not be O(n)

Insertion Sort – Analysis

- Running time depends on not only the size of the array but also the contents of the array.
- ***Best-case:*** **O(n)**
  - Array is already sorted in ascending order.
  - Inner loop will not be executed.
  - The number of moves: 2*(n-1) O(n)
  - The number of key comparisons: (n-1) O(n)
- ***Worst-case:*** **$O(n^2)$**
  - Array is in reverse order:
  - Inner loop is executed i-1 times, for i = 2,3, …, n
  - The number of moves: 2*(n-1)+(1+2+...+n-1)= 2*(n-1)+ n*(n-1)/2 $O(n^2)$
  - The number of key comparisons: (1+2+...+n-1)= n*(n-1)/2 $O(n^2)$
- Average-case: **$O(n^2)$**
  - We have to look at all possible initial data organizations.
- **So, Insertion Sort is $O(n^2)$**

Analysis of Insertion sort

- Which running time will be used to characterize this algorithm?
  - Best, worst or average?
- Worst:
  - Longest running time (this is the upper limit for the algorithm)
  - It is guaranteed that the algorithm will not be worse than this.
- Sometimes we are interested in average case. But there are some problems with the average case.
  - It is difficult to figure out the average case. i.e. what is average input?
  - Are we going to assume all possible inputs are equally likely?
  - In fact for most algorithms average case is same as the worst case.

Bubble Sort

- The list is divided into two sublists: sorted and unsorted.
- The smallest element is bubbled from the unsorted list and moved to the sorted sublist.
- After that, the wall moves one element ahead, increasing the number of sorted elements and decreasing the number of unsorted ones.
- Each time an element moves from the unsorted part to the sorted part one sort pass is completed.
- Given a list of n elements, bubble sort requires up to n-1 passes to sort the data.

Bubble Sort

23	78	45	8	32	56	Original List
23	45	8	32	56	78	After pass 1
23	8	32	45	56	78	After pass 2
8	23	32	45	56	78	After pass 3
8	23	32	45	56	78	After pass 4
8	23	32	45	56	78	After pass 5

Bubble Sort Algorithm

```
void bubbleSort(Item a[], int n)
{
    sorted = false;

   for (int i = 0; (i < n-1 && !sorted) ; i++)
    {
      sorted = true;
      m=n;
      for (int j=0; j < m; j++)
         if (a[j] > a[j+1])
         {
            swap(a[j],a[j+1]);
            sorted = false;
         }
    }
}
```

Bubble Sort

23	78	45	8	32	56	Original List
8	23	78	45	32	56	After pass 1
8	23	32	78	45	56	After pass 2
8	23	32	45	78	56	After pass 3
8	23	32	45	56	78	After pass 4
8	23	32	45	56	78	After pass 5

Bubble Sort Algorithm

```
void bubbleSort(Item a[], int n)
{
    sorted = false;

   for (int i = 0; (i < n-1 && !sorted) ; i++)
    {
      sorted = true;
      for (int j= n-1; j > i; j--)
         if (a[j-1] > a[j])
         {
            swap(a[j],a[j-1]);
            sorted = false;
         }
    }
}
```

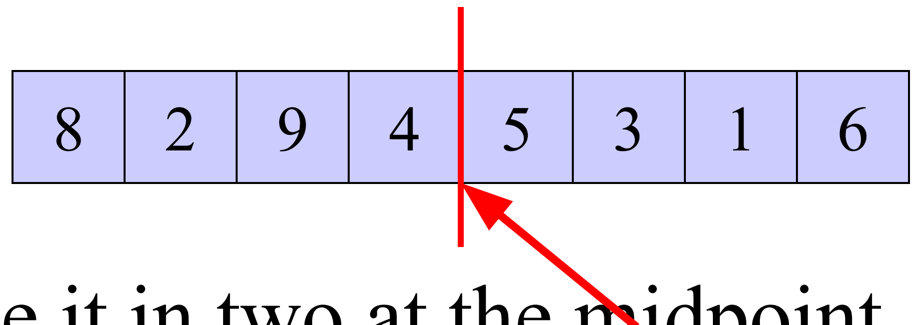
Bubble Sort – Analysis

- ***Best-case:*** □ **O(n)**
  - Array is already sorted in ascending order.
  - The number of moves: 0 □ O(1)
  - The number of key comparisons: (n-1) □ O(n)
- ***Worst-case:*** □ **$O(n^2)$**
  - Array is in reverse order:
  - Outer loop is executed n-1 times,
  - The number of moves: 3*(1+2+...+n-1) = 3 * n*(n-1)/2 □ $O(n^2)$
  - The number of key comparisons: (1+2+...+n-1)= n*(n-1)/2 □ $O(n^2)$
- Average-case: □ **$O(n^2)$**
  - We have to look at all possible initial data organizations.
- **So, Bubble Sort is $O(n^2)$**

Mergesort

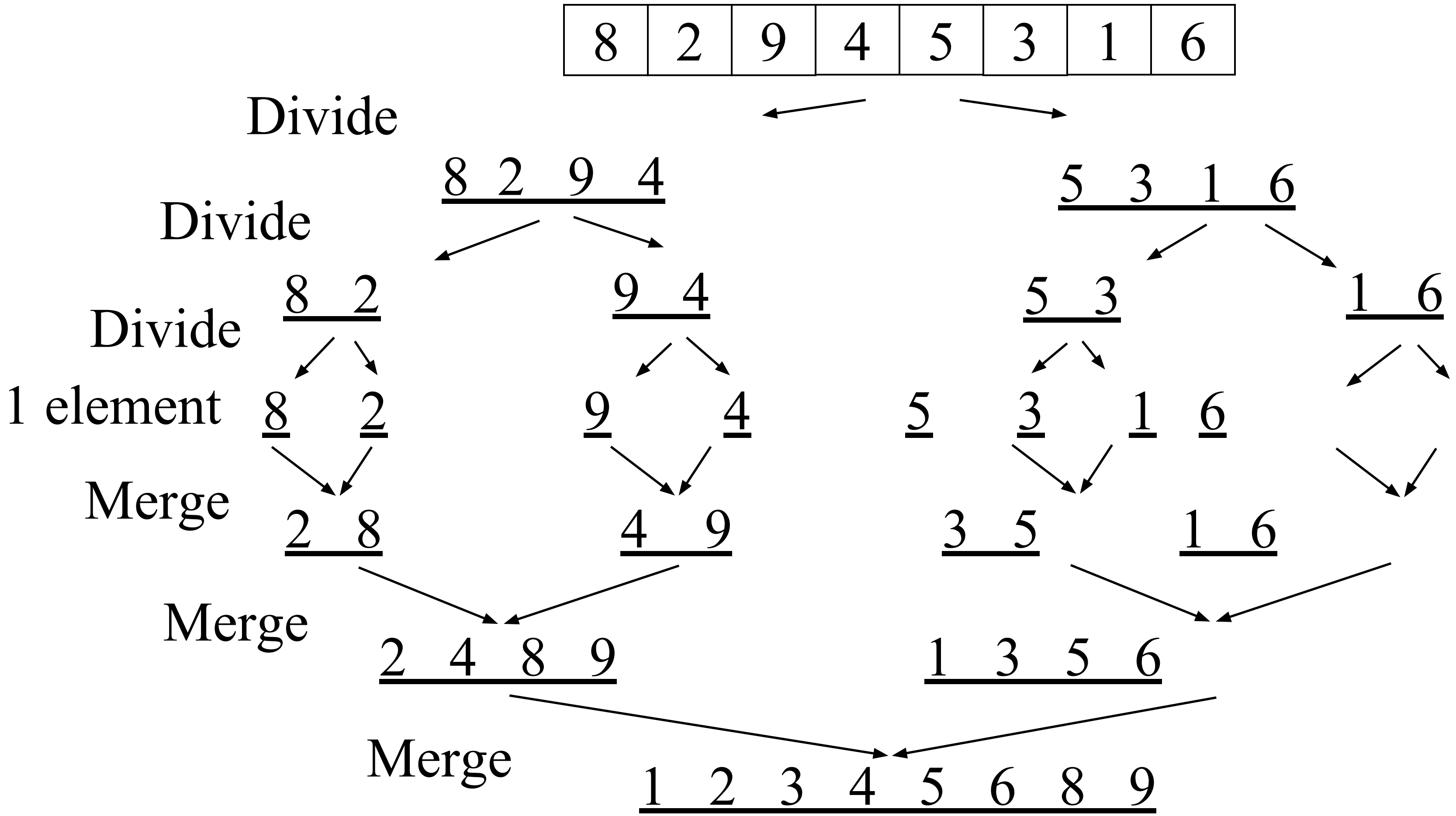
- Mergesort algorithm is one of two important divide-and-conquer sorting algorithms (the other one is quicksort).
- It is a recursive algorithm.
  - Divides the list into halves,
  - Sort each halves separately, and
  - Then merge the sorted halves into one sorted array.

Mergesort



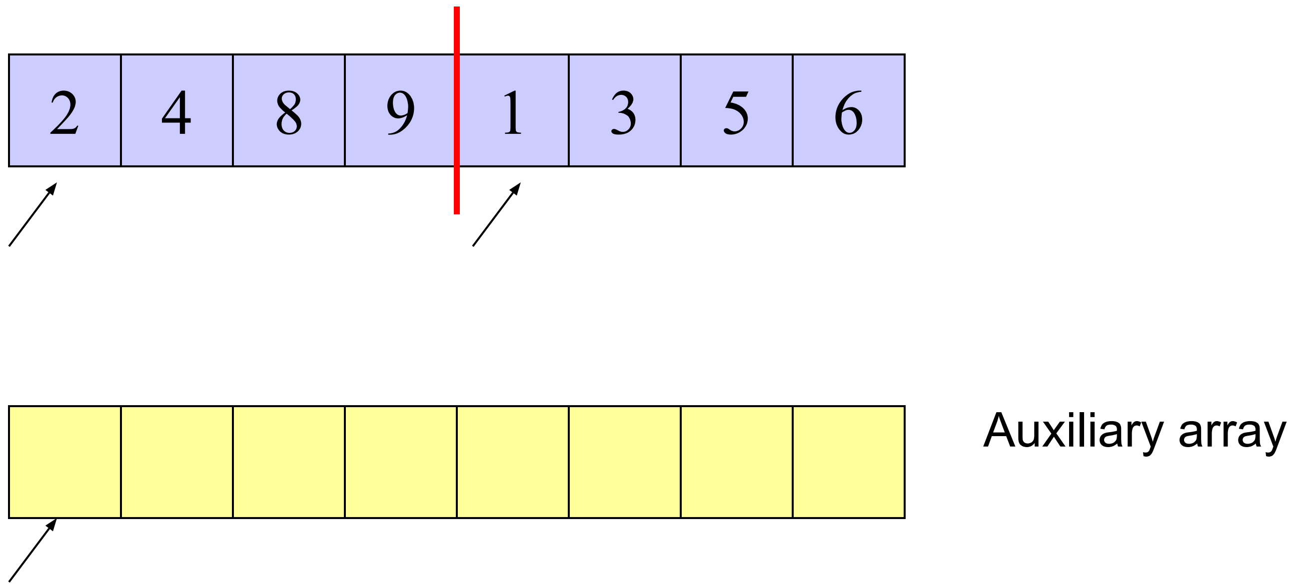
- Divide it in two at the midpoint
- Conquer each side in turn (by recursively sorting)
- Merge two halves together

Mergesort Example



Auxiliary Array

- The merging requires an auxiliary array.



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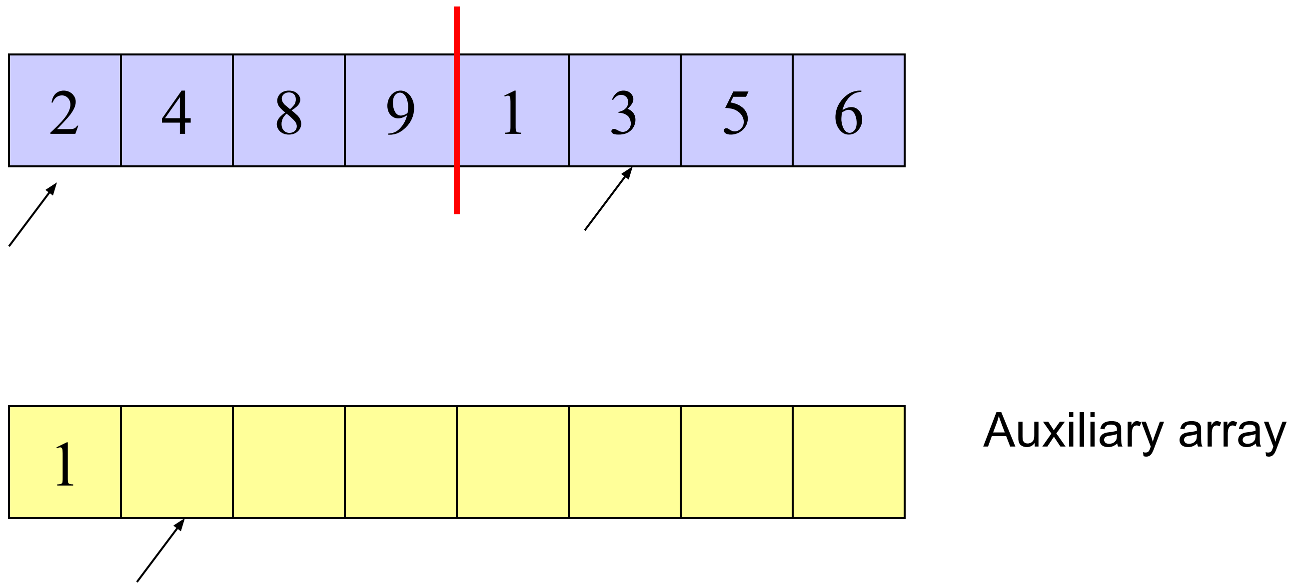

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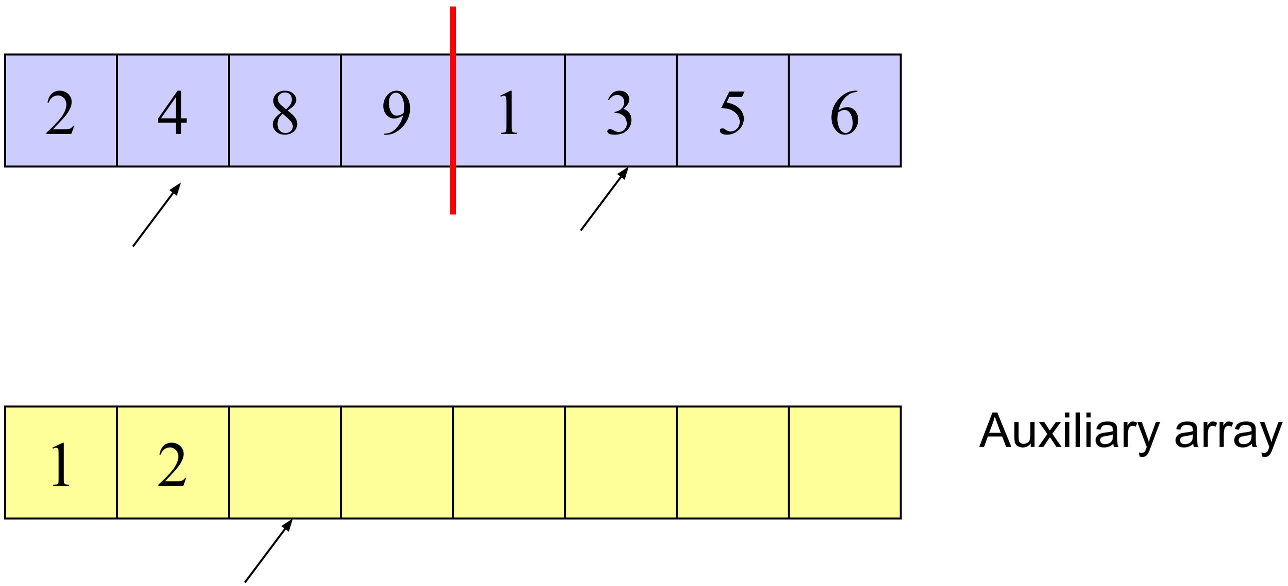

Auxiliary Array

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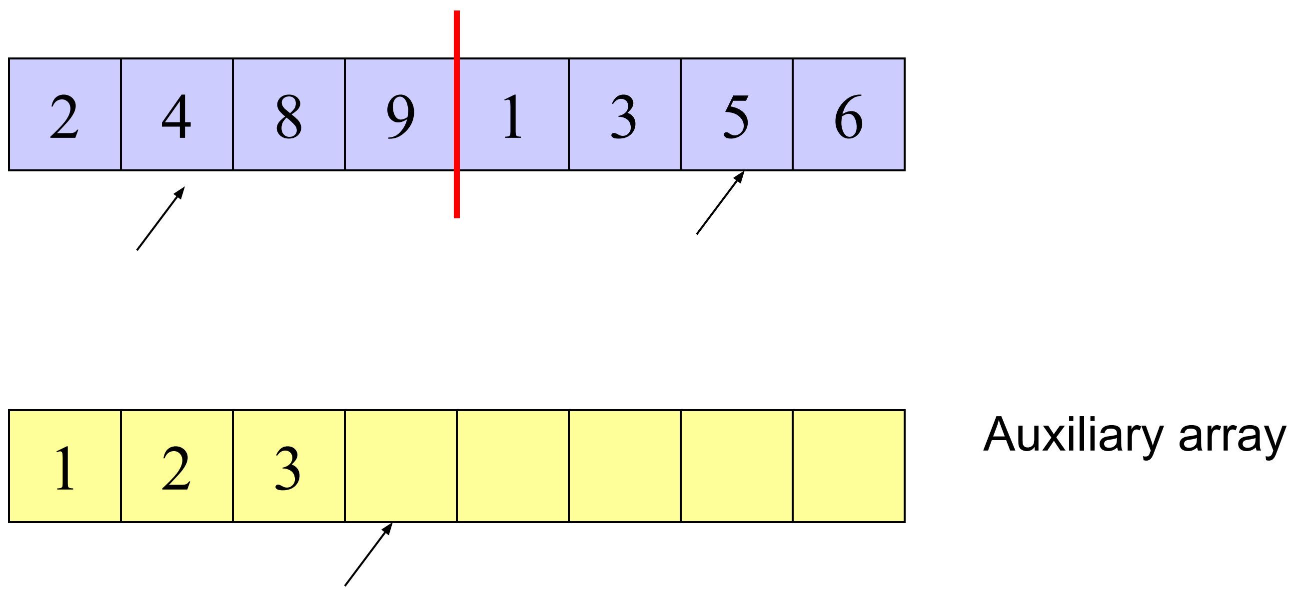

Auxiliary Array

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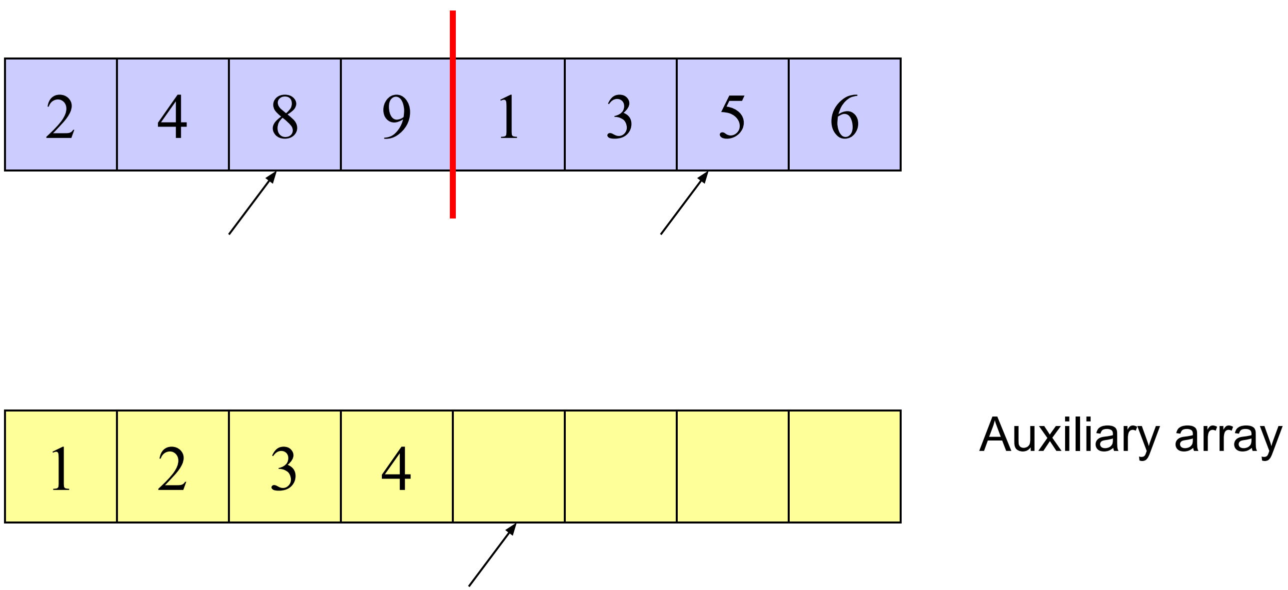

Auxiliary Array

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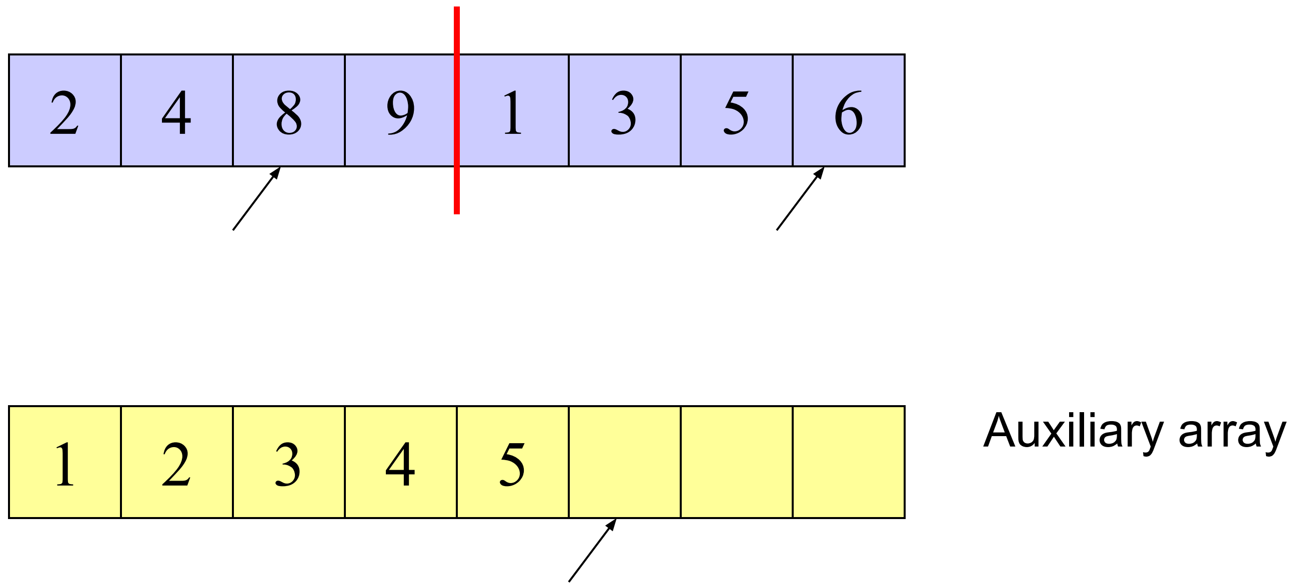

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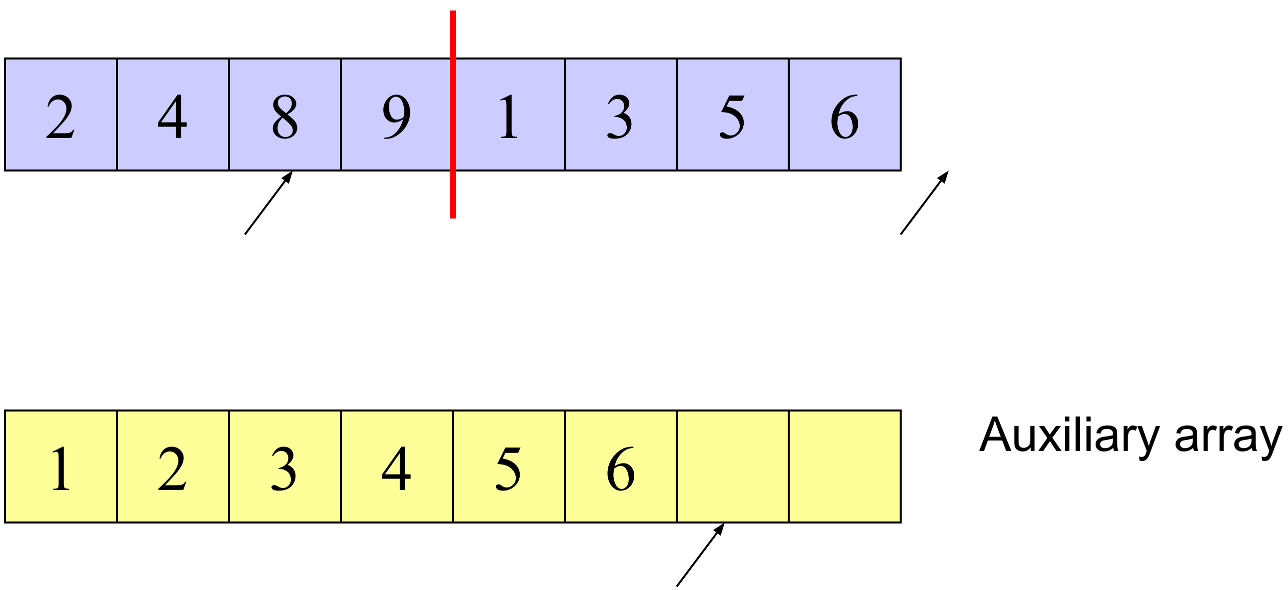

Auxiliary Array

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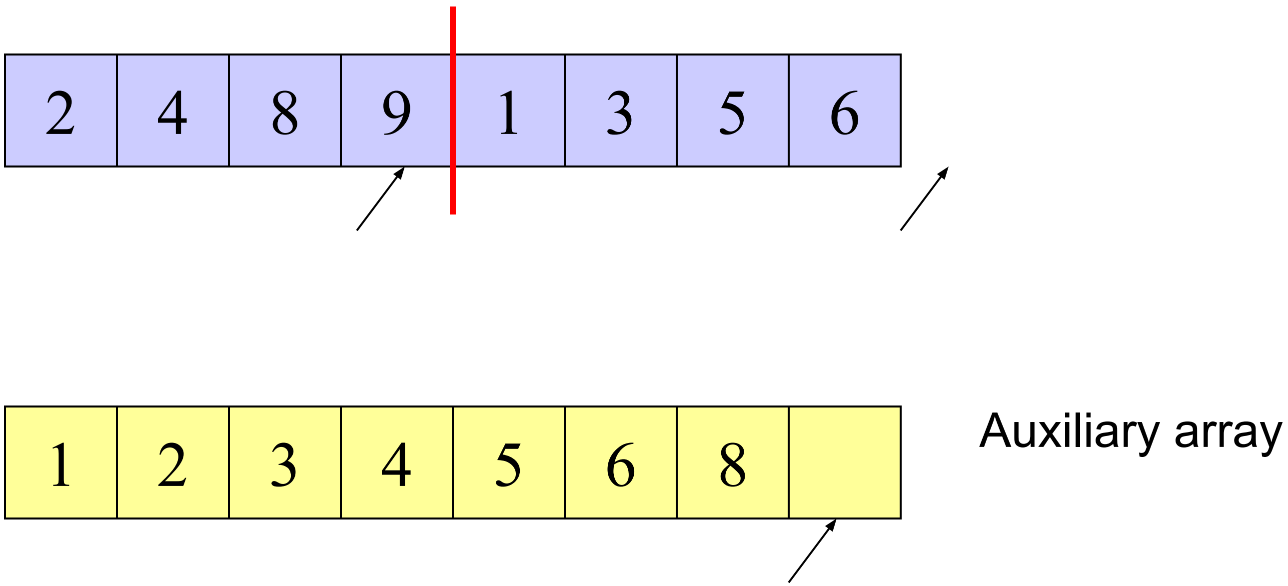

Auxiliary Array

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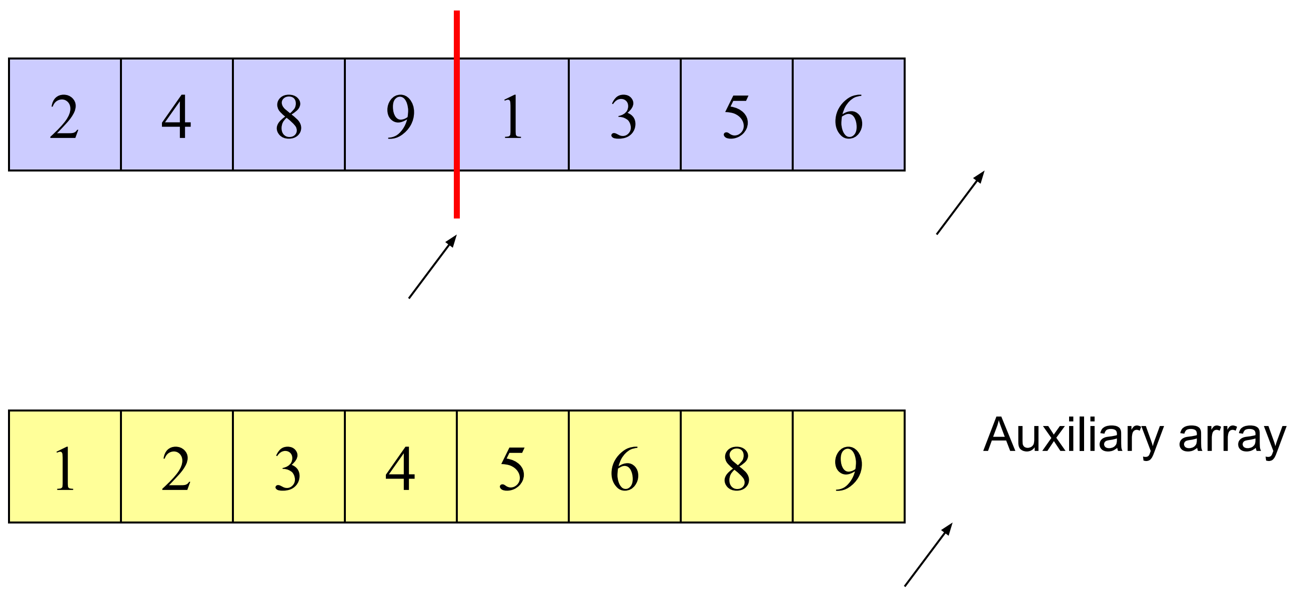
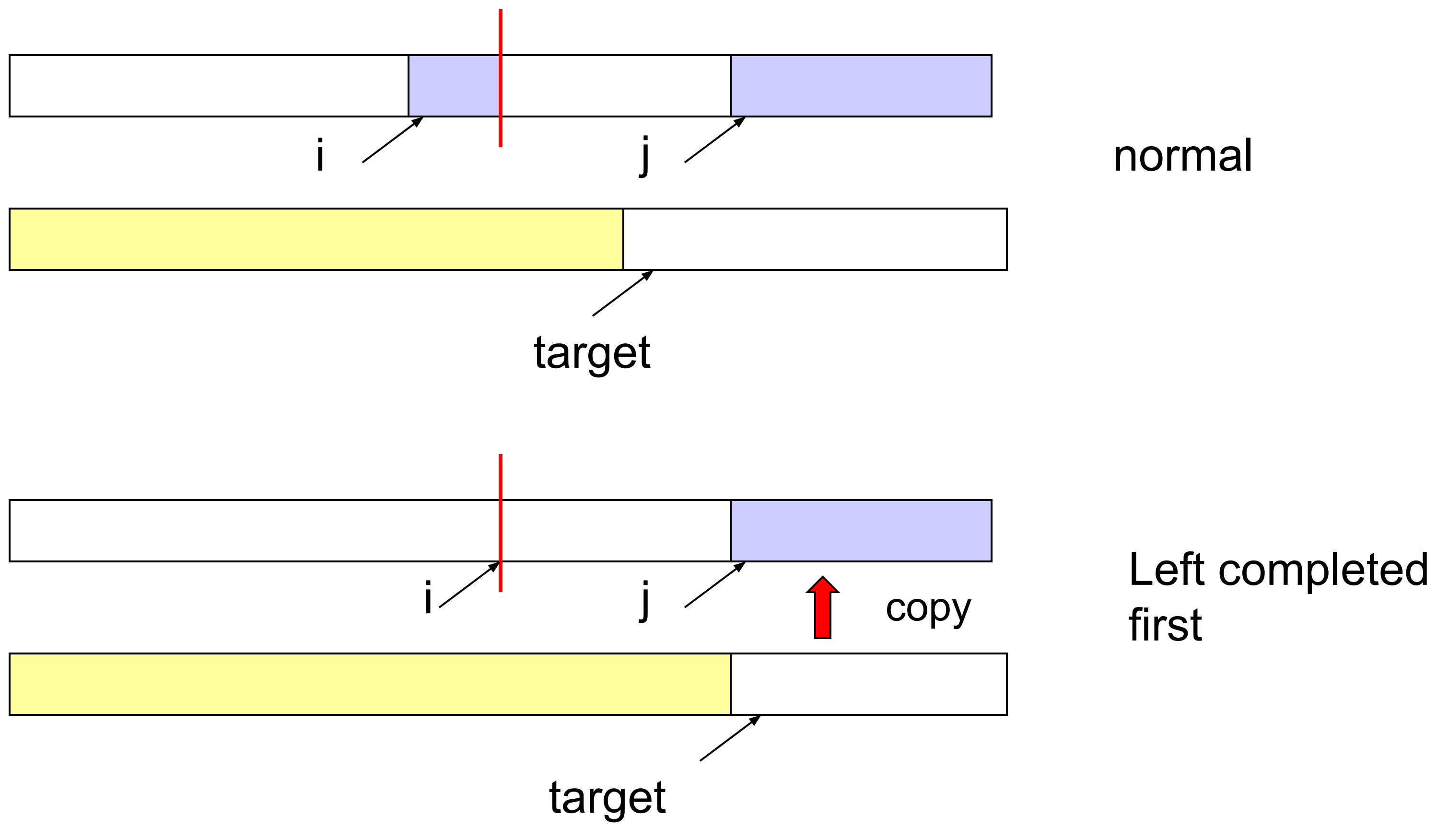
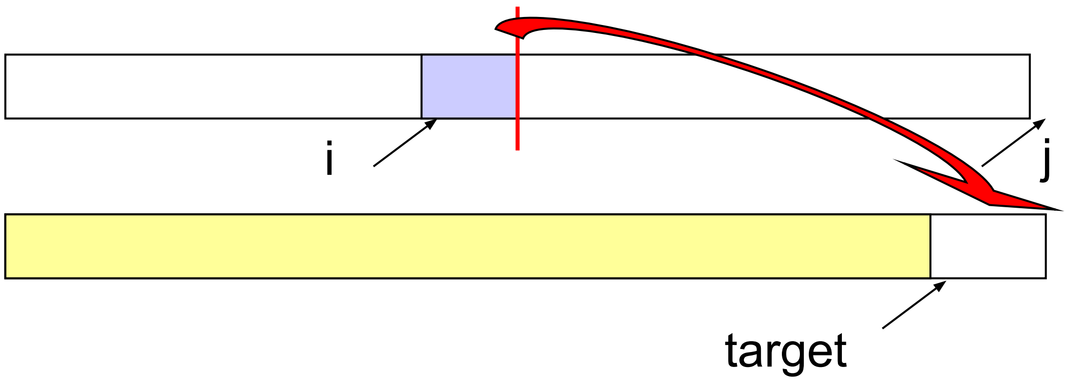


Merging



Merging



Right completed first

Merge

```
const int MAX_SIZE = maximum-number-of-items-in-array;

void merge(int Array[], int first, int mid, int last)
 {
   DataType tempArray[MAX_SIZE];      // temporary array
   int L1 = first;                     // beginning of first subarray
   int U1 = mid;                       // end of first subarray
   int L2 = mid + 1;                   // beginning of second subarray
   int U2 = last;                      // end of second subarray
   int index = first;                  // next available location in tempArray
   for (      ; (L1 <= U1) && (L2 <= U2);   index++)
    {
     if (Array[L1] < Array[L2])
     {  tempArray[index] = Array[L1];
       L1++;  }
     else
     {   tempArray[index] = Array[L2];
        L2++; }
    }
```

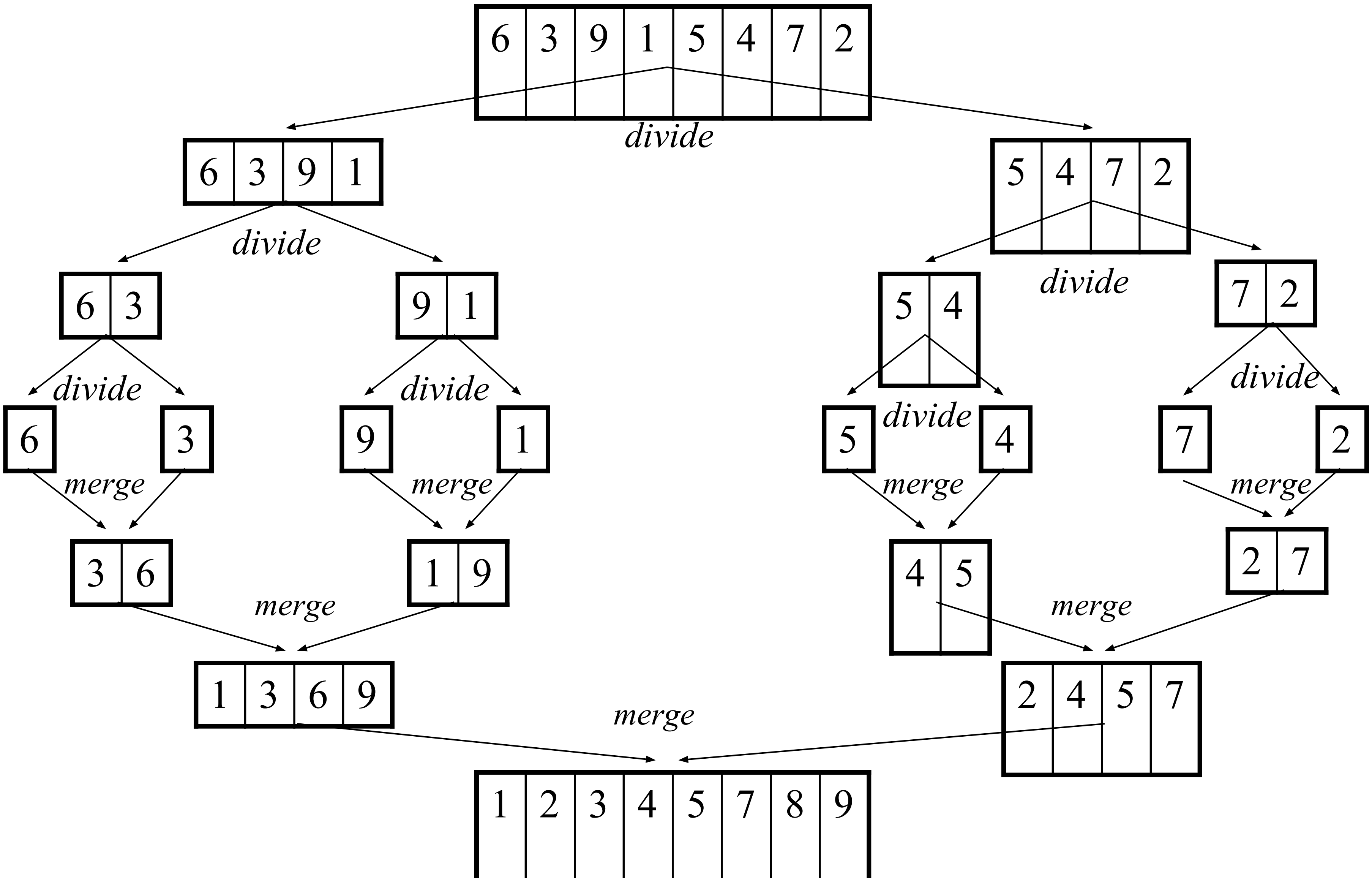
Merge (cont.)

```
                    // finish off the L subarray, if necessary
for (; L1 <= U1; L1++, index++)
  tempArray[index] = Array[L1];

                    // finish off the second subarray, if necessary
for (; L2 <= U2; L2++, index++)
  tempArray[index] = Array[L2];

                    // copy the result back into the original array
for (index = first; index <= last; index++)
  Array[index] = tempArray[index];

}  // end merge
```

Mergesort

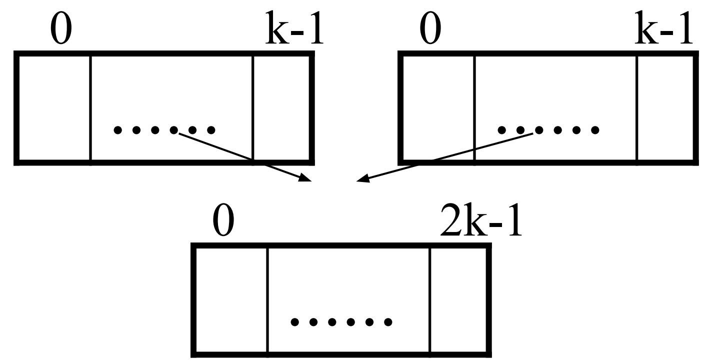
```
void mergesort(int Array[], int first, int last)
{
  if (first < last)
  {
    int mid = (first + last)/2;
    mergesort(Array, first, mid);
    mergesort(Array, mid+1, last);

    // merge the two halves
    merge(Array, first, mid, last);
  }
}
```

Mergesort - Example



Mergesort – Analysis of Merge (cont.)

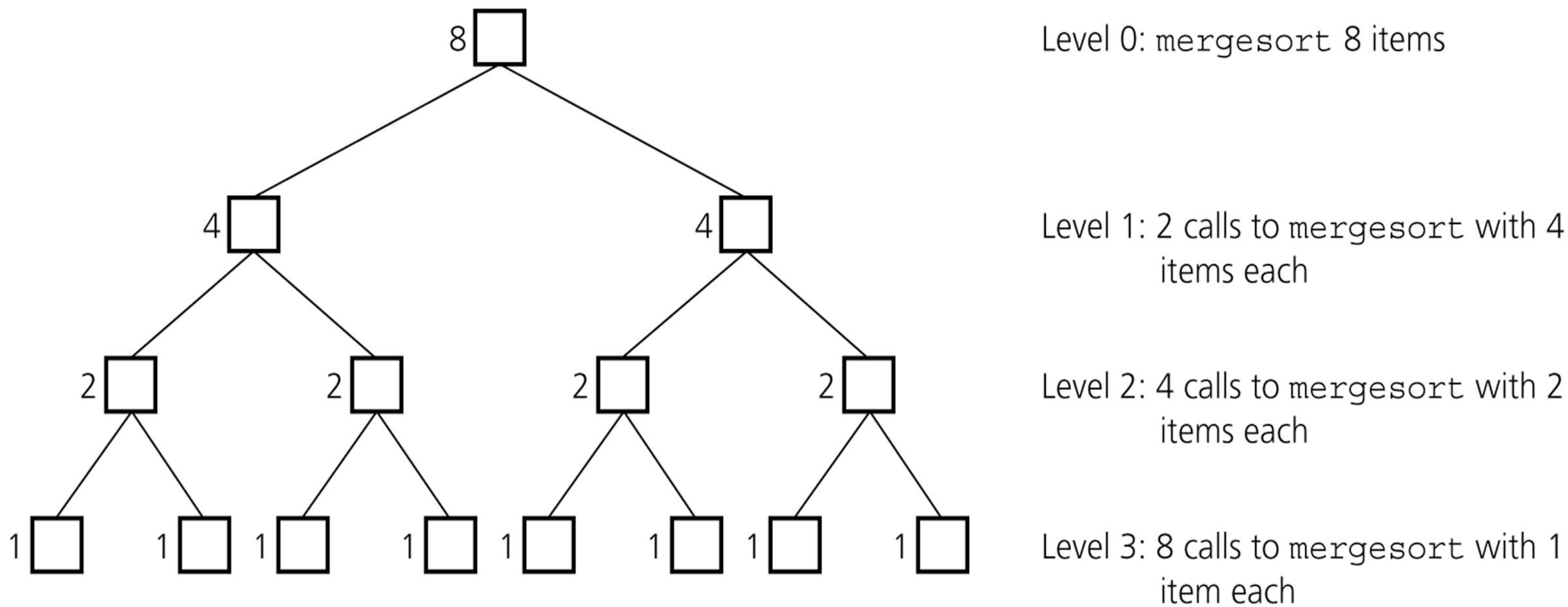
Merging two sorted arrays of size ***k***



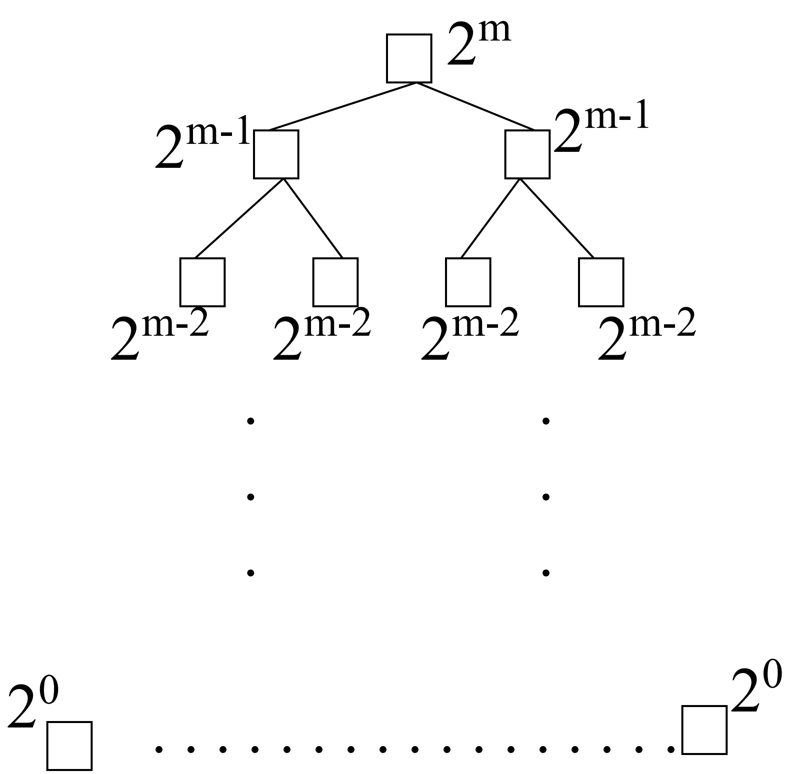
- ***Best-case:***
  - All the elements in the first array are smaller (or larger) than all the elements in the second array.
  - The number of moves: 2k + 2k
  - The number of key comparisons: k
- ***Worst-case:***
  - The number of moves: 2k + 2k
  - The number of key comparisons: 2k-1

Mergesort - Analysis

Levels of recursive calls to *mergesort*, given an array of eight items



Mergesort - Analysis



level 0 : 1 merge (size 2^{m-1})

level 1 : 2 merges (size 2^{m-2})

level 2 : 4 merges (size 2^{m-3})

level m-1 : 2^{m-1} merges (size 2^0)

level m

Mergesort - Analysis

- ***Worst-case –***

The number of key comparisons:

$= 2^0*(2*2^{m-1}-1) + 2^1*(2*2^{m-2}-1) + ... + 2^{m-1}*(2*2^0-1)$

$= (2^m - 1) + (2^m - 2) + ... + (2^m – 2^{m-1})$ (m terms)

$= m*2^m - \sum_{i=0}^{m-1} 2^i$

$= m*2^m – 2^m – 1$

Using m = log n

$= n * \log_2 n – n – 1$

□ **$O(n * \log_2 n)$**

Mergesort – Analysis

- Mergesort is extremely efficient algorithm with respect to time.
  - Both worst case and average cases are **O (n * $\log_2 n$)**

- But, mergesort requires an extra array whose size equals to the size of the original array.

- If we use a linked list, we do not need an extra array
  - But, we need space for the links
  - And, it will be difficult to divide the list into half (O(n))

Quicksort

- Like mergesort, Quicksort is also based on the *divide-and-conquer* paradigm.
- But it uses this technique in a somewhat opposite manner, as all the hard work is done *before* the recursive calls.
- It works as follows:
  1. First, it partitions an array into two parts,
  2. Then, it sorts the parts independently,
  3. Finally, it combines the sorted subsequences by a simple concatenation.

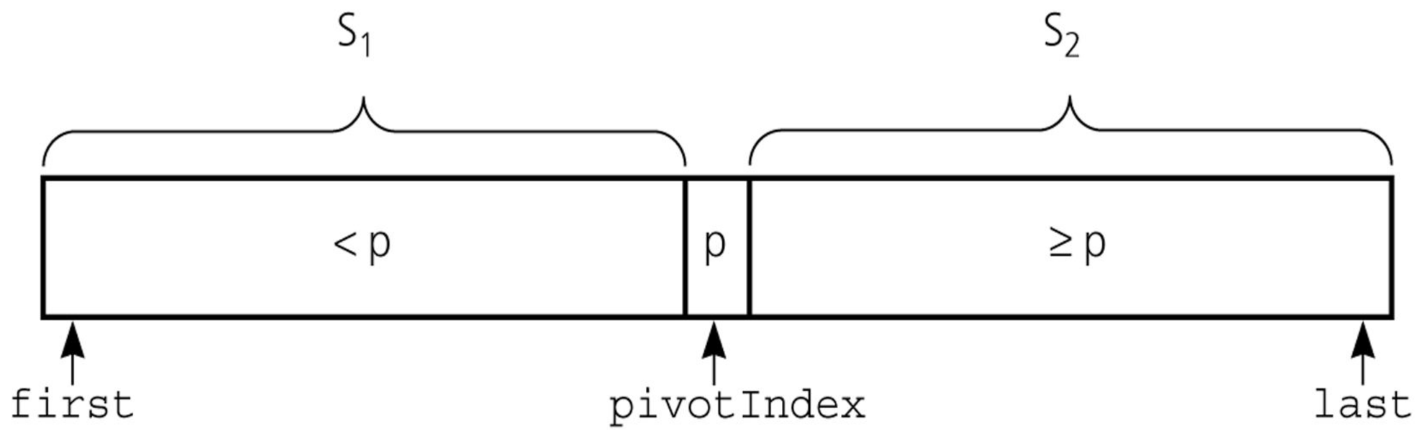
Quicksort (cont.)

The quick-sort algorithm consists of the following three steps:

1. ***Divide***: Partition the list.
   - To partition the list, we first choose some element from the list for which we hope about half the elements will come before and half after. Call this element the ***pivot***.
   - Then we partition the elements so that all those with values less than the pivot come in one sublist and all those with greater values come in another.
2. ***Recursion***: Recursively sort the sublists separately.
3. ***Conquer***: Put the sorted sublists together.

Partition

- Partitioning places the pivot in its correct place position within the array.



- Arranging the array elements around the pivot p generates two smaller sorting problems.
  - sort the left section of the array, and sort the right section of the array.
  - when these two smaller sorting problems are solved recursively, our bigger sorting problem is solved.

Partition – Choosing the pivot

- First, we have to select a pivot element among the elements of the given array, and we put this pivot into the first location of the array before partitioning.
- Which array item should be selected as pivot?
  - Somehow we have to select a pivot, and we hope that we will get a good partitioning.
  - If the items in the array arranged randomly, we choose a pivot randomly.
  - We can choose the first or last element as a pivot (it may not give a good partitioning).
  - We can use different techniques to select the pivot.

Partition Function

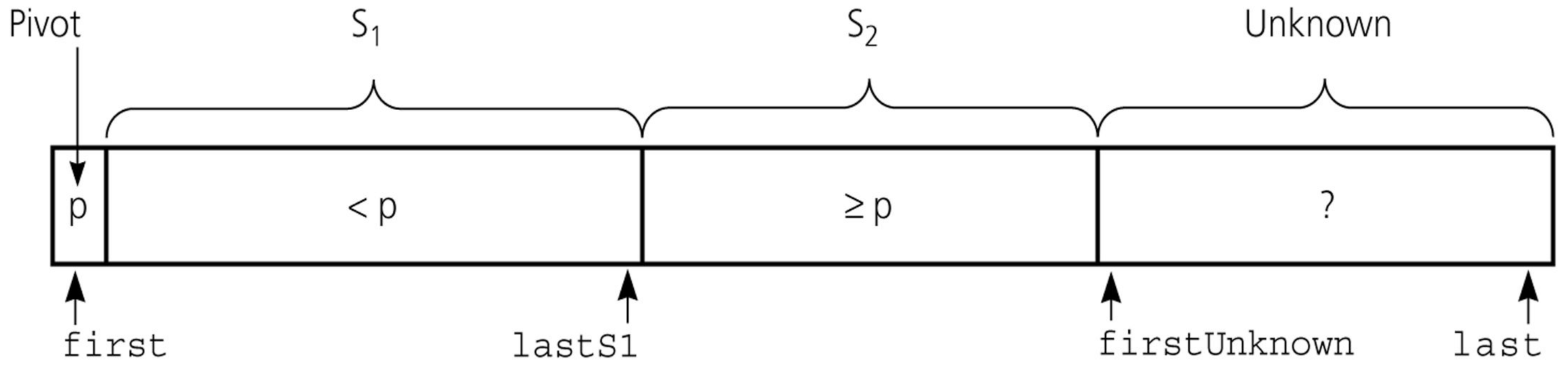
```
template <class DataType>
void partition(DataType theArray[], int first, int last,
         int &pivotIndex) {
// Partitions an array for quicksort.
// Precondition: first <= last.
// Postcondition: Partitions theArray[first..last] such that:
//    S1 = theArray[first..pivotIndex-1] < pivot
//    theArray[pivotIndex] == pivot
//    S2 = theArray[pivotIndex+1..last] >= pivot
// Calls: choosePivot and swap.

// place pivot in theArray[first]
   choosePivot(theArray, first, last);

   DataType pivot = theArray[first]; // copy pivot
```

Partition Function (cont.)

```
   // initially, everything but pivot is in unknown
   int lastS1 = first;            // index of last item in S1
   int firstUnknown = first + 1; //index of 1st item in unknown
   // move one item at a time until unknown region is empty
   for (; firstUnknown <= last; ++firstUnknown) {
      // Invariant: theArray[first+1..lastS1] < pivot
      //            theArray[lastS1+1..firstUnknown-1] >= pivot
      // move item from unknown to proper region
      if (theArray[firstUnknown] < pivot) {     // belongs to S1
         ++lastS1;
         swap(theArray[firstUnknown], theArray[lastS1]);
      }   // else belongs to S2
   }
   // place pivot in proper position and mark its location
   swap(theArray[first], theArray[lastS1]);
   pivotIndex = lastS1;
} // end partition
```

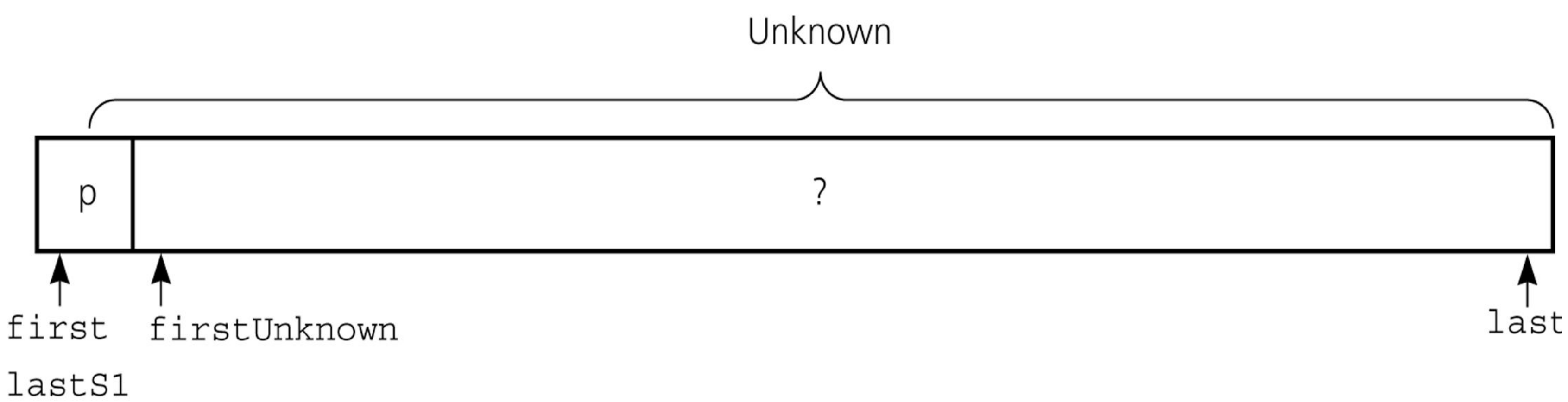
Partition Function (cont.)

Invariant for the partition algorithm



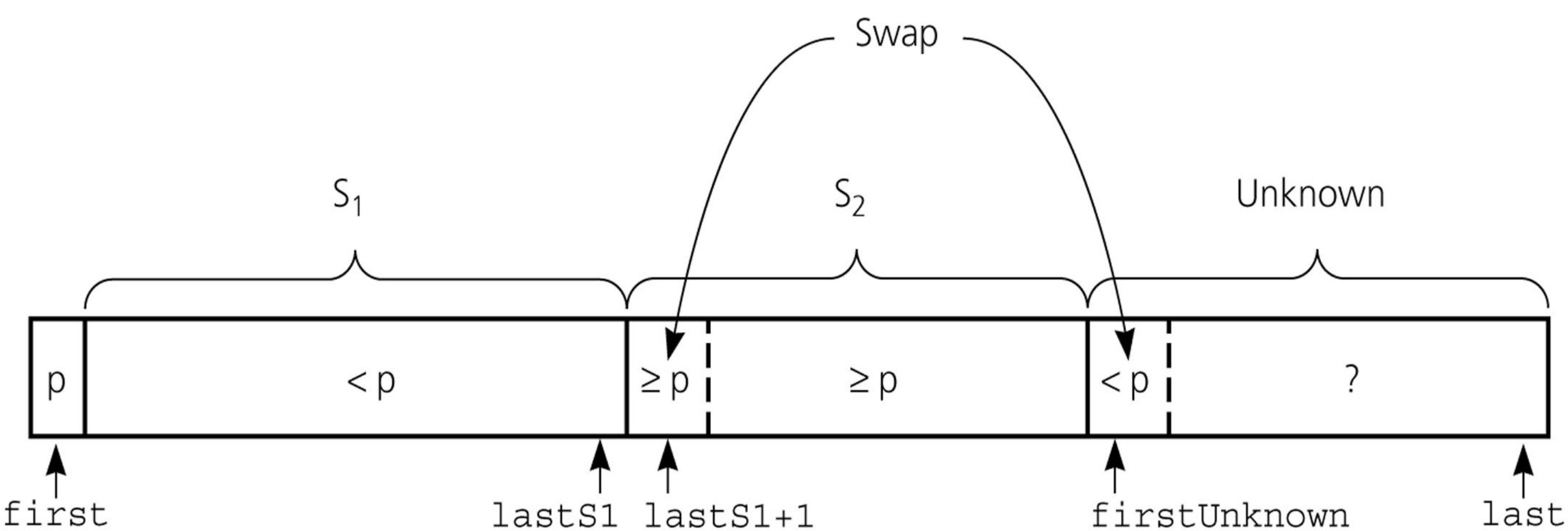
Partition Function (cont.)

Initial state of the array



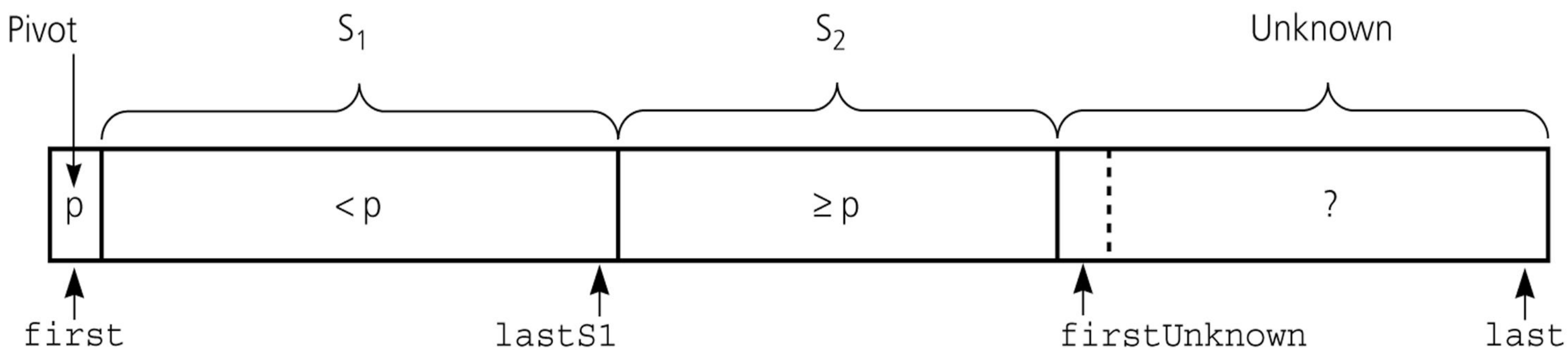
Partition Function (cont.)

Moving* `theArray[firstUnknown]` *into* S_1 *by swapping it with* `theArray[lastS1+1]` *and by incrementing both* `lastS1` *and* `firstUnknown`*.



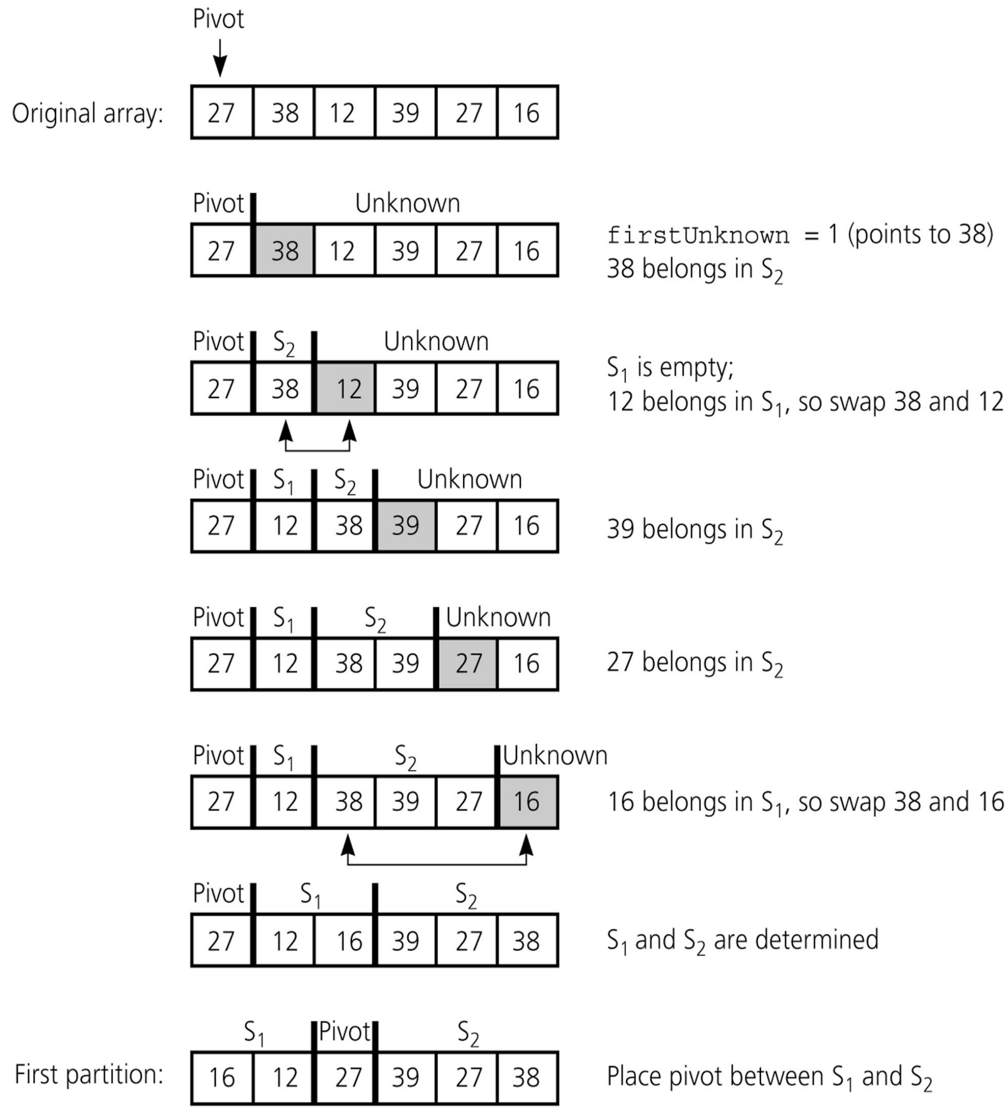
Partition Function (cont.)

Moving `theArray[firstUnknown]` into S_2 by incrementing `firstUnknown`.



Partition Function (cont.)

Developing the first partition of an array when the pivot is the first item



Quicksort Function

```
void quicksort(DataType theArray[], int first, int last) {
// Sorts the items in an array into ascending order.
// Precondition: theArray[first..last] is an array.
// Postcondition: theArray[first..last] is sorted.
// Calls: partition.
   int pivotIndex;
   if (first < last) {
      // create the partition: S1, pivot, S2
      partition(theArray, first, last, pivotIndex);
      // sort regions S1 and S2
      quicksort(theArray, first, pivotIndex-1);
      quicksort(theArray, pivotIndex+1, last);
   }
}
```

Quicksort – Analysis

Worst Case: (assume that we are selecting the first element as pivot)

– The pivot divides the list of size n into two sublists of sizes 0 and n-1.

– The number of key comparisons

$$= n\text{-}1 + n\text{-}2 + ... + 1$$

$$= \mathbf{n^2/2 - n/2} \quad □ \quad \mathbf{O(n^2)}$$

– The number of swaps =

$$= \underbrace{n\text{-}1}_{\text{swaps outside of the for loop}} + \underbrace{n\text{-}1 + n\text{-}2 + ... + 1}_{\text{swaps inside of the for loop}}$$

$$= \mathbf{n^2/2 + n/2 - 1} \quad □ \quad \mathbf{O(n^2)}$$

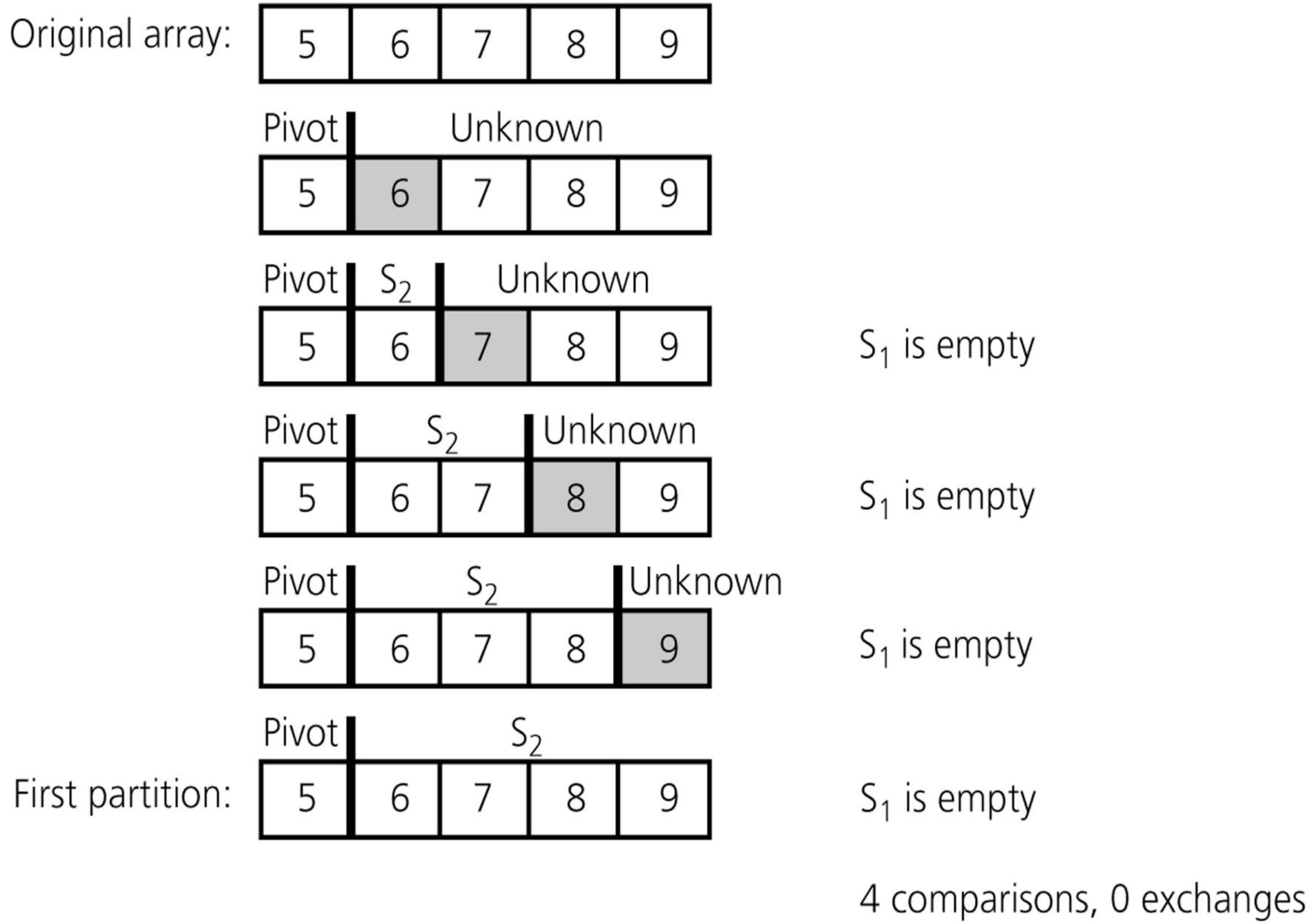
- So, Quicksort is **O(n^2)** in worst case

Quicksort – Analysis

- Quicksort is **$O(n*\log_2 n)$** in the best case and average case.
- Quicksort is slow when the array is sorted and we choose the first element as the pivot.
- Although the worst case behavior is not so good, its average case behavior is much better than its worst case.
  - So, Quicksort is one of best sorting algorithms using key comparisons.

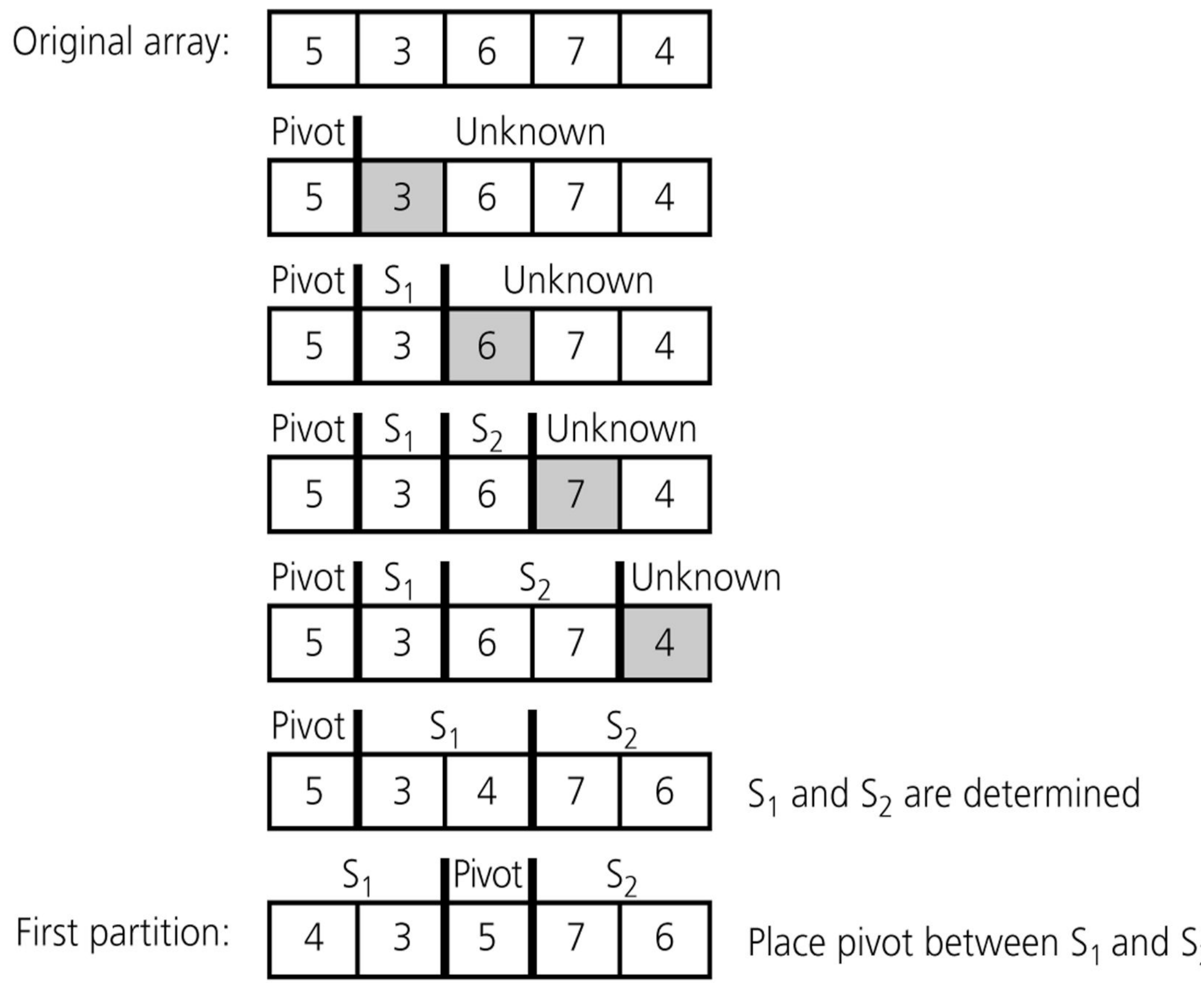
Quicksort – Analysis

A worst-case partitioning with **`quicksort`**



Quicksort – Analysis

An average-case partitioning with ***`quicksort`***



Radix Sort

- Radix sort algorithm different than other sorting algorithms that we talked.
  - It does not use key comparisons to sort an array.
- The radix sort :
  - Treats each data item as a character string.
  - First it groups data items according to their rightmost character, and put these groups into order w.r.t. this rightmost character.
  - Then, combine these groups.
  - We, repeat these grouping and combining operations for all other character positions in the data items from the rightmost to the leftmost character position.
  - At the end, the sort operation will be completed.

Radix Sort – Example

mom, dad, god, fat, bad, cat, mad, pat, bar, him	original list
(da**d**,go**d**,ba**d**,ma**d**) (mo**m**,hi**m**) (ba**r**) (fa**t**,ca**t**,pa**t**)	group strings by rightmost letter
dad,god,bad,mad,mom,him,bar,fat,cat,pat	combine groups
(d**a**d,b**a**d,m**a**d,b**a**r,f**a**t,c**a**t,p**a**t) (h**i**m) (g**o**d,m**o**m)	group strings by middle letter
dad,bad,mad,bar,fat,cat,pat,him,god,mom	combine groups
(**b**ad,**b**ar) (**c**at) (**d**ad) (**f**at) (**g**od) (**h**im) (**m**ad,**m**om) (**p**at)	group strings by first letter
bad,bar,cat,dad,fat,god,him,mad,mom,par	combine groups (SORTED)

Radix Sort – Example

0123, 2154, 0222, 0004, 0283, 1560, 1061, 2150	Original integers
(156**0**, 215**0**) (106**1**) (022**2**) (012**3**, 028**3**) (215**4**, 000**4**)	Grouped by fourth digit
1560, 2150, 1061, 0222, 0123, 0283, 2154, 0004	Combined
(00**0**4) (02**2**2, 01**2**3) (21**5**0, 21**5**4) (15**6**0, 10**6**1) (02**8**3)	Grouped by third digit
0004, 0222, 0123, 2150, 2154, 1560, 1061, 0283	Combined
(0**0**04, 1**0**61) (0**1**23, 2**1**50, 2**1**54) (0**2**22, 0**2**83) (1**5**60)	Grouped by second digit
0004, 1061, 0123, 2150, 2154, 0222, 0283, 1560	Combined
(**0**004, **0**123, **0**222, **0**283) (**1**061, **1**560) (**2**150, **2**154)	Grouped by first digit
0004, 0123, 0222, 0283, 1061, 1560, 2150, 2154	Combined (sorted)

Radix Sort - Algorithm

```
radixSort(inout theArray:ItemArray, in n:integer, in d:integer)
// sort n d-digit integers in the array theArray
    for (j=d down to 1) {
        Initialize 10 groups to empty
        Initialize a counter for each group to 0
        for (i=0 through n-1) {
            k = jth digit of theArray[i]
            Place theArrar[i] at the end of group k
            Increase kth counter by 1
        }
        Replace the items  in theArray with all the items in group 0,
        followed by all the items in group 1, and so on.
    }
```

Radix Sort -- Analysis

- The radix sort algorithm requires 2*n*d moves to sort n strings of d characters each.
  - So, Radix Sort is **O(n)**

- Although the radix sort is O(n), it is not appropriate as a general-purpose sorting algorithm.
  - Its memory requirement is ***d * original size of data*** (because each group should be big enough to hold the original data collection.)
  - For example, we need 27 groups to sort string of uppercase letters.
  - The radix sort is more appropriate for a linked list than an array. (we will not need the huge memory in this case)

Comparison of Sorting Algorithms

	Worst case	Average case
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Mergesort	$n * \log n$	$n * \log n$
Quicksort	n^2	$n * \log n$
Radix sort	n	n
Treesort	n^2	$n * \log n$
Heapsort	$n * \log n$	$n * \log n$