SOFT COMPUTING
MODULE 3

FUZZY LOGIC

Topics

- Fuzzy Logic
- Fuzzy Sets
- Properties of fuzzy set
- Operations on fuzzy set
- Fuzzy Relations
- Operations on fuzzy relations

Fuzzy Logic

- Is an essential component of Soft Computing
- Meaning of fuzzy -not clear ,noisy etc
- Concept of fuzzy logic was conceived by Lotfi Zadeh
- Fuzzy Logic is a multivalued logic which allow intermediate values in between conventional true/false ,yes/no, black/white etc.
- Boolean logic is a two-valued logic deals with only true/false ,yes/no, black/white

- Fuzzy logic is a problem solving methodology implemented in systems from simple ,small embedded microcontrollers to complex, large netwoked multichannel PC
- Fuzzy logic can be implemented in hardware,software or a combination of both
- FL is a way to arrive at a conclusion based upon vague, ambiguous, imprecise , noisy or missing information

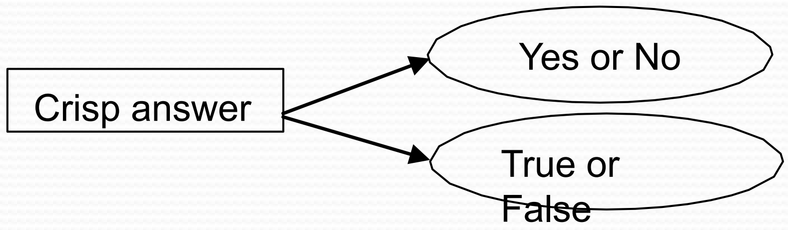
History of Fuzzy Logic

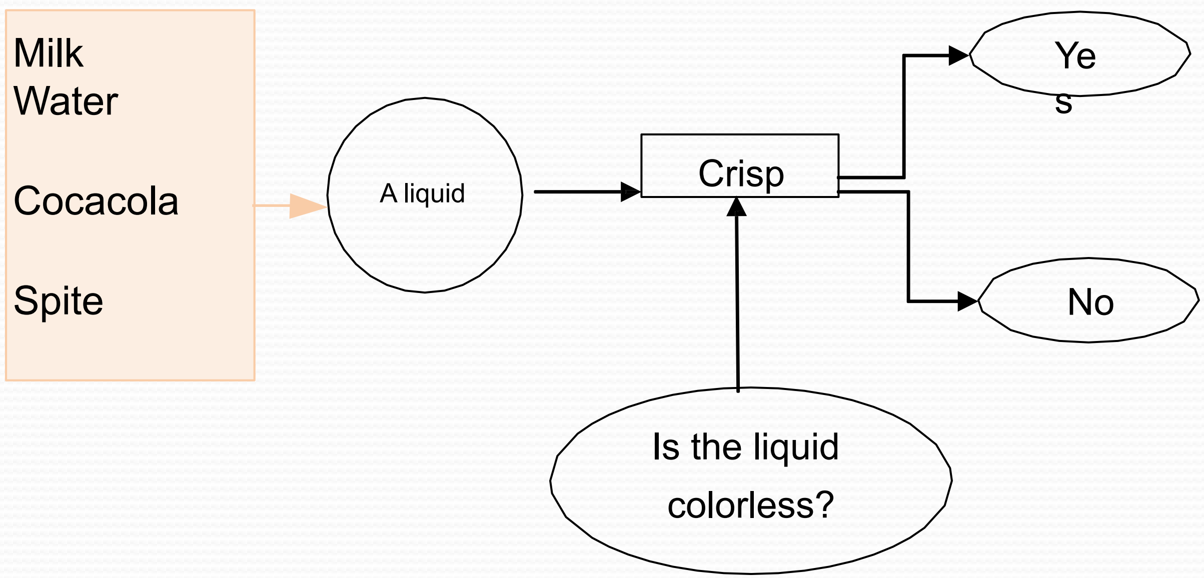
- Fuzzy set thory was introduced by Professor Lotfi Zadeh (USA) in 1965 as an extension of the classical set thory
- 1972 First working group on fuzzy systems in Japan by Toshiro Terano
- 1973 A paper on fuzzy algorithms by Zadeh (USA)
- 1974 Steam engine control by Ebrahim Mamdani (UK)
- Too many events, inventions and projects to mention till 1991
- After 1991 fuzzy technology came out of scientific laboratories and became an industrial tool.
- In the last two decades, the fuzzy sets theory has established itself as a new methodology for dealing with any sort of ambiguity and uncertainty.

Fuzzy Sets

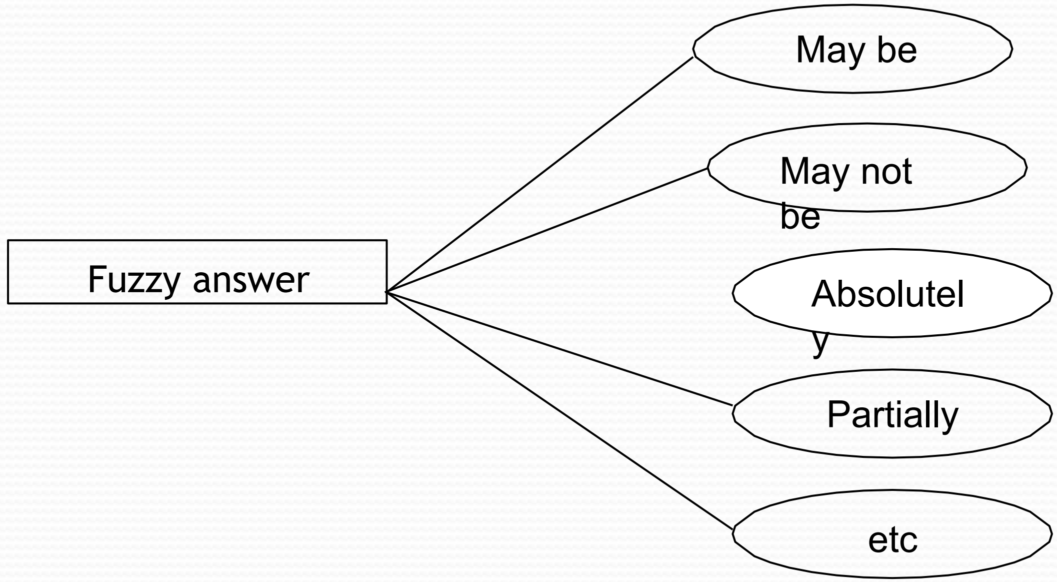
- A classical *set X* is a collection of definite, distinguishable elements. Each element can either belong to not belong to a set.
- A crisp (classical) set is a set for which each value is either included or not included in the set.
- For a fuzzy set, every element has a membership value, and so is a member to some extent.
- The ***membership value*** defines the extent to which a variable is a member of a fuzzy set.
- The membership value is from 0 (not at all a member of the set) to 1.

Fuzzy logic vs. Crisp logic

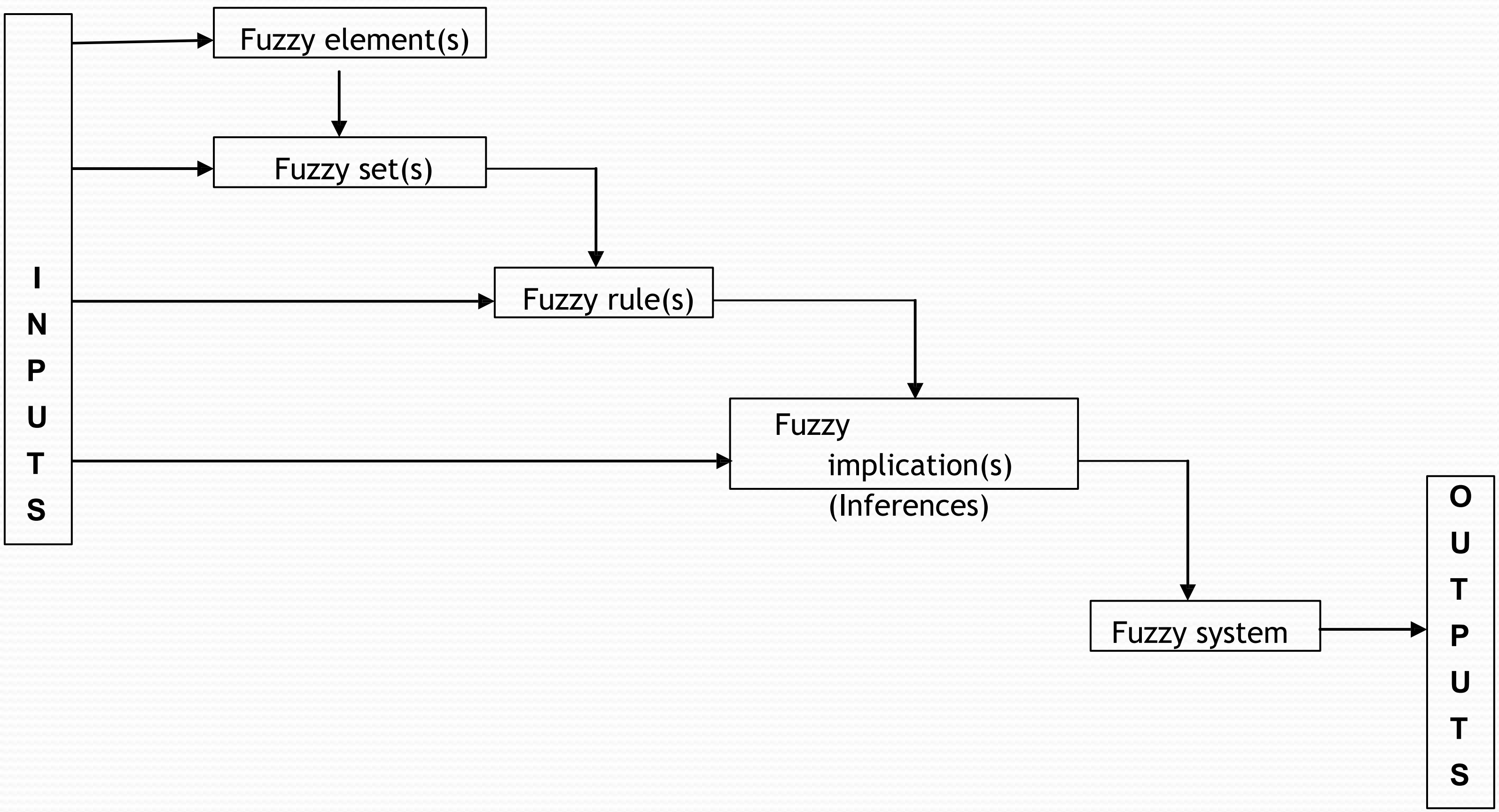




Fuzzy logic vs. Crisp logic



Concept of fuzzy system



- Fuzzy system has many ingredients or elements
- One or more fuzzy element Fuzzy set
- Many fuzzy sets+ fuzzy elements Fuzzy rule
- Set of fuzzy rules govern something Fuzzy inference
-

Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

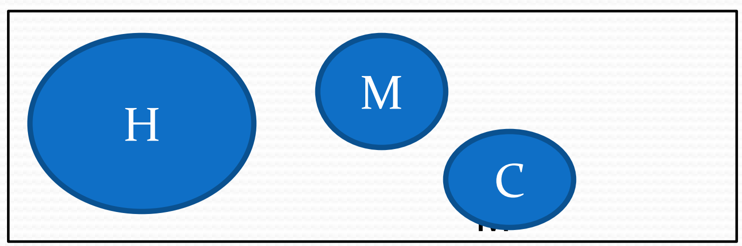
X = The entire population of India.

H = All Hindu population = $\{ h_1, h_2, h_3, \ldots , h_L \}$

M = All Muslim population = $\{ m_1, m_2, m_3, \ldots , m_N \}$

C = All Christian population = $\{ c_1, c_2, c_3, \ldots , c_N \}$

Universe of discourse **X**



Here, All are the sets of finite numbers of individuals. Such a set is called crisp set.

Refreshing Crisp sets

Classical sets(Crisp sets)

- A *set* is defined as a collection of objects, which share certain characteristics.
- A *classical set* is a collection of distinct objects.
- Each individual entity in a set is called a *member* or an *element* of the set.
- Collection of elements in the universe*(U)* is called *whole set*.
- Number of elements in *U* is called *cardinal number*.
- Collection of elements within a set are called *subsets*.
- Classical set is defined as the *U* is splitted in to two groups:
- *members and nonmembers*.
- No partial membership exists

- There are different ways for defining a crisp or a classical set:

1. The list of all the members of a set may be given.

$$A = \{2, 4, 6, 8, 10\}$$

2. The properties of the set elements may be specified.

$$A = \{x \mid x \text{ is prime number} < 20\}$$

3. The formula for the definition of a set may be mentioned.

$$A = \left\{x = \frac{n}{n+1}, \text{ where n is a natural number less than 6}\right\}$$

4. The set may be defined on the basis of the results of a logical operation.

$$A = \{x \mid x \text{ is an element belonging to } P \text{ AND } Q\}$$

5. There exist a membership function, which may also be used to define a set.

$$\mu_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Operations on Classical Sets

1. Union

$$A \cup B = \{x | x \in A \ or \ x \in B\}$$

2. Intersection

$$A \cap B = \{x | x \in A \ and \ x \in B\}$$

3. Complement

$$\overline{A} = \{x | x \notin A, \ x \in X\}$$

4. Difference(Subtraction)

$$A | B \ or \ (A - B) = \{x | x \in A \ and \ x \notin B\} = A - (A \cap B)$$

$$B | A \ or \ (B - A) = \{x | x \in B \ and \ x \notin A\} = B - (B \cap A)$$

Problems

(1) *Find the power set and cardinality of the given set* $X = \{2, 4, 6\}$. *Also find cardinality of power set.*

Set X contains 3 elements, so,

$$n_X = 3$$

The power set of X is,

$$P(X) = \{\phi, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}, \{2,4,6\}\}$$

The cardinality of power set P(X) is,

$$n_{P(X)} = 2^{n_X} = 2^3 = 8$$

Example 12 Let A = { 2, 4, 6, 8} and B = { 6, 8, 10, 12}. Find $A \cup B$.

Solution We have $A \cup B = \{ 2, 4, 6, 8, 10, 12\}$

Note that the common elements 6 and 8 have been taken only once while writing $A \cup B$.

Example 13 Let A = { *a, e, i, o, u* } and B = { *a, i, u* }. Show that $A \cup B = A$

Solution We have, $A \cup B = \{ a, e, i, o, u \} = A$.

This example illustrates that union of sets A and its subset B is the set A itself, i.e., if $B \subset A$, then $A \cup B = A$.

Example 17 Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and B = { 2, 3, 5, 7 }. Find A ∩ B and hence show that A ∩ B = B.

Solution We have A ∩ B = { 2, 3, 5, 7 } = B. We note that B ⊂ A and that A ∩ B = B.

Example 20 Let U = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$. Find A'.

Solution We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A. Hence $A' = \{2, 4, 6, 8, 10\}$.

Example 18 Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A.

Solution We have, A – B = { 1, 3, 5 }, since the elements 1, 3, 5 belong to A but not to B and B – A = { 8 }, since the element 8 belongs to B and not to A.

We note that A – B ≠ B – A.

Properties of Classical Sets

- Commutativity

$$A \cup B = B \cup A; A \cap B = B \cap A$$

- Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C; A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C); A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Idempotency

$$A \cup A = A; A \cap A = A$$

- Transitivity

$$A \subseteq B \subseteq C, \textit{ then } A \subseteq C$$

- Identity

$$A \cup \phi = A; A \cap \phi = \phi$$
$$A \cup X = X; A \cap X = A$$

- Commutativity

$$A \cup B = B \cup A; A \cap B = B \cap A$$

- Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C; A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C); A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Idempotency

$$A \cup A = A; A \cap A = A$$

- Transitivity

$$A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

- Identity

$$A \cup \phi = A; A \cap \phi = \phi$$

$$A \cup X = X; A \cap X = \mathrm{A}$$

Properties contd..

- Involution

$$\overline{\overline{A}} = A$$

- Law of excluded middle

$$A \cup \overline{A} = X$$

- Law of contradiction

$$A \cap \overline{A} = \phi$$

- DeMorgan's law

$$|\overline{A \cap B}| = \overline{A} \cup \overline{B}$$
$$|\overline{A \cup B}| = \overline{A} \cap \overline{B}$$

Function Mapping of Classical Sets

- *Mapping* is a rule of correspondence between set theoretic forms and function theoretic forms.
- A classical set is represented by its characteristic function, $\chi(x)$, where x is the element in the universe.

1. *Union*$(A \cup B)$

$$\chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x) = max\{\chi_A(x), \chi_B(x)\}$$

2. *Intersection*$(A \cap B)$

$$\chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x) = min\{\chi_A(x), \chi_B(x)\}$$

3. *Complement*$(\overline{A})$

$$\chi_{\overline{A}}(x) = 1 - \chi_A(x)$$

4. *Containment*

$$If\ A \subseteq B,\ then\ \chi_A(x) \leq \chi_B(x)$$

Fuzzy Sets

- Fuzzy sets may be viewed as an extension and generalization of the basic concepts of crisp sets.
- It allows partial membership.
- A *fuzzy set* is a set having degrees of membership between *1 and 0*.
- Member of one fuzzy set can also be member of other fuzzy sets in the same universe.
- Vagueness is introduced in fuzzy set by eliminating the sharp boundaries that divide members from non–members in the group.
- *Possibility distribution:* A fuzzy set A in the universe of discourse U can be defined as a set of ordered pairs and it is given by,

$$A = \{(x, \mu_A(x)) | x \in U\}$$

Where, $\mu_A(x)$ is the degree of membership of x in A. $\mu_A(x) \in [0, 1]$

- When U is discrete and finite, fuzzy set A is given as:

$$A = \{\frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \frac{\mu_A(x_3)}{x_3} + ...\} = \{\sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}\}$$

  - n is a finite value.
  - The *summation symbol (+)* indicates the collection of each element.

- When U is continuous and infinite, fuzzy set A is given as:

$$A = \{\int \frac{\mu_A(x)}{x}\}$$

  - The *integral sign ($\int$)* is a continuous function–theoretic union for continuous variables.

- A fuzzy set is *universal fuzzy set* if and only if the value of membership function is *1* for all members.

$$\mu_U(x) = 1$$

- The universal fuzzy set can also be called *whole fuzzy set*.
- Two fuzzy sets A and B are equal if,

$$\mu_A(x) = \mu_B(x) \text{ for all } x \in U$$

- A fuzzy set A is an *empty fuzzy set* if and only if value of membership function is *0* for all members.

$$\mu_\phi(x) = 0$$

Crisp Set vs Fuzzy Set

Crisp set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu(s)) \mid s \in X$ and $\mu(s)$ is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an elememt $s \in X$ into F is fuzzy, that is, if present, then with a degree of **membership**.

Fuzzy set Representation

A fuzzy set can be expressed as a set of ordered pairs

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set

Membership function (MF)

Universe or universe of discourse

- *A fuzzy set is totally characterized by a membership function (MF).*
- *MF maps each element of X to a membership grade (or value) between 0 and one*

Alternate Notation

- A fuzzy set A can be alternatively denoted as follows:

X is discrete ⇒ $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous ⇒ $$A = \int_X \mu_A(x) / x$$

- **Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.**
- Crisp Sets ≤ Fuzzy Sets or in other words, Crisp Sets are Special cases of Fuzzy Sets

Example of Fuzzy set Representation

- A= { (x_1,0.8), (x_2, 0.3), (x_3, 0.1), (x_4, 0.9) }
- Can be represented in another way as
- A= $0.8/x_1 + 0.3/x_2 + 0.1/x_3 + 0.9/x_4$

Example (Discrete Universe)

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$ —— # courses a student may take in a semester.

$$A = \begin{Bmatrix} (1, 0.1) & (2, 0.3) & (3, 0.8) & (4, 1) \\ (5, 0.9) & (6, 0.5) & (7, 0.2) & (8, 0.1) \end{Bmatrix}$$ —— appropriate # courses taken

Alternate Representation:

$$A = 0.1/1 + 0.3/2 + 0.8/3 + 1.0/4 + 0.9/5 + 0.5/6 + 0.2/7 + 0.1/8$$

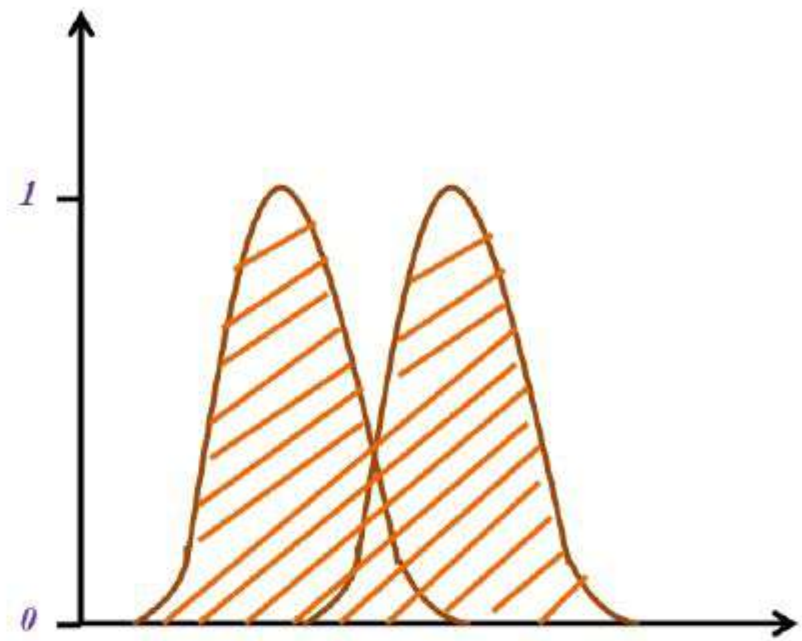
Basic Fuzzy Set Operations

- Union
- Intersection
- Complement
- Difference

Fuzzy Set Operations

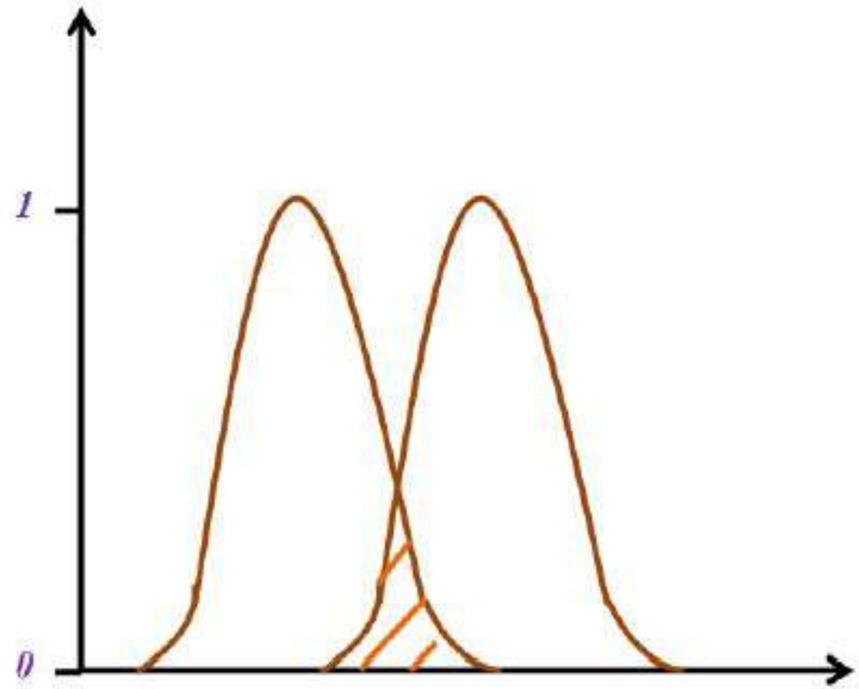
- *Union*

$$\mu_{A \cup B}(x) = max[\mu_A(x), \mu_B(x)] = \mu_A(x) \vee \mu_B(x), \text{ for all } x \in U$$



- *Intersection*

$$\mu_{A \cap B}(x) = min[\mu_A(x), \mu_B(x)] = \mu_A(x) \bigcap \mu_B(x), \text{ for all } x \in U$$



- *Complement*

$$\mu_{\overline{A}} = 1 - \mu_A(x), \; for \; all \; x \in U$$

- *Difference*

$$\mu_{A-B}(x) = min[\ \mu_A(x),\ \mu_{\overline{B}}(x)]$$

$$\mu_{B-A}(x) = min[\ \mu_B(x),\ \mu_{\overline{A}}(x)]$$

Problems

Example

Consider two fuzzy sets *A* and *B*. *F*ind Complement, Union, Intersection

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.2}{5} + \frac{0.6}{6} \right\},$$

$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{0.3}{6} \right\}.$$

Solution...

Complement

$$\bar{\underset{\sim}{A}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.4}{4} + \frac{0.8}{5} + \frac{0.4}{6} \right\},$$

$$\bar{\underset{\sim}{B}} = \left\{ \frac{0.5}{2} + \frac{0.2}{3} + \frac{0.6}{4} + \frac{0.3}{5} + \frac{0.7}{6} \right\}.$$

Union

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \left\{ \frac{1}{2} + \frac{0.8}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0.6}{6} \right\}$$

Maximum is used

…….Solution

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.2}{5} + \frac{0.6}{6} \right\},$$

$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{0.3}{6} \right\}.$$

intersection

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.4}{4} + \frac{0.2}{5} + \frac{0.3}{6} \right\}.$$

Minimum is used

(2) Consider 2 given fuzzy sets,

$$A = \{\frac{1}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0.2}{8}\}$$
$$B = \{\frac{0.5}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{1}{8}\}$$

Perform,

(a) Union

(b) Intersection

(c) Complement

(d) Difference

(a) Union

$$A \cup B = max\{\mu_A(x), \mu_B(x)\}$$
$$= \{\frac{1}{2} + \frac{0.4}{4} + \frac{0.5}{6} + \frac{1}{8}\}$$

(b) Intersection

$$A \cap B = min\{\mu_A(x), \mu_B(x)\}$$
$$= \{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.1}{6} + \frac{0.2}{8}\}$$

(c) Complement

$$\overline{A} = 1 - \mu_A(x) = \{\frac{0}{2} + \frac{0.7}{4} + \frac{0.5}{6} + \frac{0.8}{8}\}$$
$$\overline{B} = 1 - \mu_B(x) = \{\frac{0.5}{2} + \frac{0.6}{4} + \frac{0.9}{6} + \frac{0}{8}\}$$

(d) ***Difference***

$$A|B = A \cap \overline{B} = \{\frac{0.5}{2} + \frac{0.3}{4} + \frac{0.5}{6} + \frac{0}{8}\}$$

$$B|A = B \cap \overline{A} = \{\frac{0}{2} + \frac{0.4}{4} + \frac{0.1}{6} + \frac{0.8}{8}\}$$

(3) Consider 2 given fuzzy sets,

$$B_1 = \{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\}$$
$$B_2 = \{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\}$$

Perform,

$(a) B_1 \cup B_2$

$(b) B_1 \cap B_2$

$(c) \overline{B_1}$

$(d) \overline{B_2}$

$(e) B_1 | B_2$

$(f) \overline{B_1 \cup B_2}$

$(g) \overline{B_1 \cap B_2}$

$(h) B_1 \cap \overline{B_1}$

$(i) B_1 \cup \overline{B_1}$

$(j) B_2 \cap \overline{B_2}$

$(k) B_2 \cup \overline{B_2}$

$$(a)\ B_1 \cup B_2 = \{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\}$$

$$(b)\ B_1 \cap B_2 = \{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\}$$

$$(c)\ \overline{B_1} = \{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\}$$

$$(d)\ \overline{B_2} = \{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\}$$

$$(e)\ B_1 | B_2 = B_1 \cap \overline{B_2} = \{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\}$$

$$(f)\ \overline{B_1 \cup B_2} = \overline{B_1} \cap \overline{B_2} = \{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\}$$

$$(g)\ \overline{B_1 \cap B_2} = \overline{B_1} \cup \overline{B_2} = \{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\}$$

$$(h)\ B_1 \cap \overline{B_1} = \{\frac{0}{1.0} + \frac{0.25}{1.5} + \frac{0.3}{2.0} + \frac{0.15}{2.5} + \frac{0}{3.0}\}$$

$$(i)\ B_1 \cup \overline{B_1} = \{\frac{1}{1.0} + \frac{0.75}{1.5} + \frac{0.7}{2.0} + \frac{0.85}{2.5} + \frac{1}{3.0}\}$$

$$(j)\ B_2 \cap \overline{B_2} = \{\frac{0}{1.0} + \frac{0.4}{1.5} + \frac{0.2}{2.0} + \frac{0.1}{2.5} + \frac{0}{3.0}\}$$

$$(k)\ B_2 \cup \overline{B_2} = \{\frac{1}{1.0} + \frac{0.6}{1.5} + \frac{0.8}{2.0} + \frac{0.9}{2.5} + \frac{1}{3.0}\}$$

(4) It is necessary to compare two sensors based upon their detection levels and gain settings. The table of gain settings and sensor detection levels with a standard item being monitored providing typical membership values to represent the detection levels for each sensor is given in table:

Gain setting	Detection level of sensor 1	Detection level of sensor 2
0	0	0
10	0.2	0.35
20	0.35	0.25
30	0.65	0.8
40	0.85	0.95
50	1	1

Perform union, intersection, complement and difference over sensor 1 and sensor 2.

Given the universe of discourse,

$$X = \{0, 10, 20, 30, 40, 50\}$$

The membership functions for the two sensors in the discrete form as,

$$D_1 = \{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\}$$
$$D_2 = \{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\}$$
$$D_1 \Longrightarrow Sensor1$$
$$D_2 \Longrightarrow Sensor2$$

(a) Union

$$D_1 \cup D_2 = max[D_1, D_2] = \{\frac{0}{0} + \frac{0.35}{10} + \frac{0.35}{20} + \frac{0.8}{30} + \frac{0.95}{40} + \frac{1}{50}\}$$

(b) Intersection

$$D_1 \cap D_2 = min[D_1, D_2] = \{\frac{0}{0} + \frac{0.2}{10} + \frac{0.25}{20} + \frac{0.65}{30} + \frac{0.85}{40} + \frac{1}{50}\}$$

(c) Complement

$$\overline{D_1} = 1 - D_1 = \{\frac{1}{0} + \frac{0.8}{10} + \frac{0.65}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\}$$

$$\overline{D_2} = 1 - D_2 = \{\frac{1}{0} + \frac{0.65}{10} + \frac{0.75}{20} + \frac{0.2}{30} + \frac{0.05}{40} + \frac{0}{50}\}$$

(d) Difference

$$D_1 | D_2 = D_1 \cap \overline{D_2} = \{\frac{0}{0} + \frac{0.2}{10} + \frac{0.35}{20} + \frac{0.2}{30} + \frac{0.05}{40} + \frac{0}{50}\}$$

$$D_2 | D_1 = D_2 \cap \overline{D_1} = \{\frac{0}{0} + \frac{0.35}{10} + \frac{0.25}{20} + \frac{0.35}{30} + \frac{0.15}{40} + \frac{0}{50}\}$$

5. Design a computer software to perform image processing to locate objects within a scene. The two fuzzy sets representing a plane and a train image are:

$$\text{Plane} = \left\{ \frac{0.2}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.1}{\text{house}} \right\}$$

$$\text{Train} = \left\{ \frac{1}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.2}{\text{house}} \right\}$$

Find the following:

(a) $Plane \cup Train$; (b) $Plane \cap Train$;

(c) $\overline{Plane}$; (d) $\overline{Train}$;

(e) $Plane | Train$; (f) $\overline{Plane \cup Train}$;

(g) $\overline{Plane \cap Train}$; (h) $Plane \cup \overline{Plane}$;

(i) $Plane \cap \overline{Plane}$; (j) $Train \cup \overline{Train}$;

(k) $Train \cup \overline{Train}$

(a) $\underset{\sim}{\text{Plane}} \cup \underset{\sim}{\text{Train}}$

$$= \max\{\mu_{\underset{\sim}{\text{Plane}}}(x), \mu_{\underset{\sim}{\text{Train}}}(x)\}$$

$$= \left\{\frac{1.0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.2}{\text{house}}\right\}$$

(b) $\underset{\sim}{\text{Plane}} \cap \underset{\sim}{\text{Train}}$

$$= \min\{\mu_{\underset{\sim}{\text{Plane}}}(x), \mu_{\underset{\sim}{\text{Train}}}(x)\}$$

$$= \left\{\frac{0.2}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

(c) $\overline{\underset{\sim}{\text{Plane}}} = 1 - \mu_{\underset{\sim}{\text{Plane}}}(x)$

$$= \left\{ \frac{0.8}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.9}{\text{house}} \right\}$$

(d) $\overline{\underset{\sim}{\text{Train}}} = 1 - \mu_{\underset{\sim}{\text{Train}}}(x)$

$$= \left\{ \frac{0}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.8}{\text{house}} \right\}$$

(e) $\text{Plane} \,|\, \text{Train}$

$$= \text{Plane} \cap \overline{\text{Train}}$$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{ \frac{0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.1}{\text{house}} \right\}$$

(f) $\overline{\text{Plane} \cup \text{Train}}$

$$= 1 - \max\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{ \frac{0}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.8}{\text{house}} \right\}$$

(g) $\overline{\text{Plane} \cap \text{Train}}$

$$= 1 - \min\{\mu_{\text{Plane}}(x), \mu_{\text{Train}}(x)\}$$

$$= \left\{\frac{0.8}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.9}{\text{house}}\right\}$$

(h) $\text{Plane} \cup \overline{\text{Plane}}$

$$= \max\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$

$$= \left\{\frac{0.8}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.7}{\text{boat}} + \frac{0.8}{\text{plane}} + \frac{0.9}{\text{house}}\right\}$$

(i) $\text{Plane} \cap \overline{\text{Plane}}$

$$= \min\{\mu_{\text{Plane}}(x), \mu_{\overline{\text{Plane}}}(x)\}$$

$$= \left\{\frac{0.2}{\text{train}} + \frac{0.5}{\text{bike}} + \frac{0.3}{\text{boat}} + \frac{0.2}{\text{plane}} + \frac{0.1}{\text{house}}\right\}$$

(j) $\underset{\sim}{\text{Train}} \cup \overline{\underset{\sim}{\text{Train}}}$

$$= \max\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{ \frac{1.0}{\text{train}} + \frac{0.8}{\text{bike}} + \frac{0.6}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.8}{\text{house}} \right\}$$

(k) $\underset{\sim}{\text{Train}} \cap \overline{\underset{\sim}{\text{Train}}}$

$$= \min\{\mu_{\text{Train}}(x), \mu_{\overline{\text{Train}}}(x)\}$$

$$= \left\{ \frac{0}{\text{train}} + \frac{0.2}{\text{bike}} + \frac{0.4}{\text{boat}} + \frac{0.5}{\text{plane}} + \frac{0.2}{\text{house}} \right\}.$$

More Operations on Fuzzy Sets

- *Algebraic sum*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

- *Algebraic product*

$$\mu_{A.B}(x) = \mu_A(x).\mu_B(x)$$

- *Bounded sum*

$$\mu_{A \oplus B}(x) = min[1, \mu_A(x) + \mu_B(x)]$$

- *Bounded difference*

$$\mu_{A \odot B}(x) = max[0, \mu_A(x) - \mu_B(x)]$$

(5) Consider 2 given fuzzy sets,

$$A = \{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3}\}$$

$$B = \{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3}\}$$

Find,

(a) algebraic sum

(b) algebraic product

(c) bounded sum

(d) bounded difference

(a) Algebraic sum

$$\mu_{A+B}(X) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$$
$$= \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3}\} - \{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3}\}$$
$$= \{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3}\}$$

(b) Algebraic product

$$\mu_{A \cdot B}(X) = \mu_A(x) \cdot \mu_B(x)$$
$$= \{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3}\}$$

(c) Bounded sum

$$\mu_{A \bigoplus B}(X) = min[1, \mu_A(x) + \mu_B(x)]$$
$$= min\{1, \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3}\}\}$$
$$= \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3}\}$$

(d) Bounded difference

$$\mu_{A \ominus B}(X) = max[0, \mu_A(x) - \mu_B(x)]$$
$$= max\{0, \{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3}\}\}$$
$$= \{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3}\}$$

Algebraic sum

The algebraic sum of fuzzy sets A and B is a fuzzy set C.

$$\tilde{C} = \tilde{A} + \tilde{B}$$

where $\mu_C(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\,\mu_B(x)$

Example

Let $\tilde{A} = \frac{0.5}{2} + \frac{0.7}{3} + \frac{1}{5}$

and $\tilde{B} = \frac{0.2}{2} + \frac{0.8}{5}$

then $C = A + B \quad \mu_C(x_1) = 0.5 + 0.2 - 0.1$

$= 0.7 - 0.1 = 0.6$

$\mu_C(x_2) = 0.7 + 0 - 0 = 0.7$

$\mu_C(x_3) = 1 + 0.8 - 0.8 = 1$

$$\therefore \tilde{C} = \tilde{A} + \tilde{B} = \frac{0.6}{2} + \frac{0.7}{3} + \frac{1}{5}$$

Algebraic product

The algebraic product of two fuzzy sets

Example

Let $$\tilde{A} = \frac{0.1}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{1}{5}$$

$$\tilde{B} = \frac{0.3}{4} + \frac{0.5}{5} + \frac{0.8}{6} + \frac{1}{7}$$

then $$\tilde{A}\,\tilde{B} = \frac{0.24}{4} + \frac{0.5}{5}$$

Bounded sum

The bounded sum of two fuzzy sets A and B is denoted by

$$\tilde{C} = \tilde{A} \oplus \tilde{B}$$

where $\mu_C(x) = \min[1, \mu_A(x) + \mu_B(x)]$

The bounded sum for

Let $\tilde{A} = \frac{0.5}{2} + \frac{0.7}{3} + \frac{1}{5}$

and $\tilde{B} = \frac{0.2}{2} + \frac{0.8}{5}$

$$\mu_C(x_1) = \min(1, 0.5 + 0.2) = 0.7$$

$$\mu_C(x_2) = \min(1, 0.7 + 0) = 0.7$$

$$\mu_C(x_3) = \min(1, 1 + 0.8) = 1$$

$$\therefore \quad \tilde{C} = \tilde{A} \oplus \tilde{B} = \frac{0.7}{2} + \frac{0.7}{3} + \frac{1}{5}$$

Bounded difference

The bounded difference of two fuzzy sets A and B is defined by

$$\tilde{C} = \tilde{A} \ominus \tilde{B}$$

where $\mu_C(x) = \min[1, \mu_A(x) - \mu_B(x)]$ or $\max[0, \mu_A(x) - \mu_B$

e.g $\max[0, 0.3 - 0.7] = \max[0, -0.4] = 0$

hence $0 \leq \mu_A(x) - \mu_B(x) \leq 1$

Let $\tilde{A} = \frac{0.5}{2} + \frac{0.7}{3} + \frac{1}{5}$

and $\tilde{B} = \frac{0.2}{2} + \frac{0.8}{5}$

The bounded difference for example

$$\mu_C(x_1) = \min(1, 0.5 - 0.2) = 0.3$$

$$\mu_C(x_2) = \min(1, 0.7 - 0) = 0.7$$

$$\mu_C(x_3) = \min(1, 1 - 0.8) = 0.2$$

Thus $\tilde{C} = \tilde{A} \ominus \tilde{B} = \frac{0.3}{2} + \frac{0.7}{3} + \frac{0.2}{5}$

Power of a fuzzy set A^{α}:

$$\mu_{A^{\alpha}}(x) = \{\mu_A(x)\}^{\alpha}$$

If $a < 1$, then it is called *dilation*

If $a > 1$, then it is called *concentration*

The operations concentration and dilation are defined as

$$CON(\tilde{A}) = \tilde{A}^2 \text{ and } DIL(\tilde{A}) = \tilde{A}^{0.5}$$

$$\text{Let} \quad \tilde{A} = \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.7}{2}$$

$$\text{then} \quad CON(\tilde{A}) = \frac{0.49}{2} + \frac{0.25}{3} + \frac{0.04}{4}$$

$$\text{and} \quad DIL\ \tilde{A} = \frac{0.7}{3} + \frac{0.44}{4} + \frac{0.84}{2}$$

Power of a fuzzy set

The α power of a fuzzy set A is a new set A^{α} with the MF

$$\mu_{A^{\alpha}}(x) = (\mu_A(x))^{\alpha}$$

Example

$$A = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

$$\alpha = 2$$

$$A^{\alpha} = \{(x_1, 0.16), (x_2, 0.36), (x_3, 0.64)\}$$

Product of a two fuzzy sets

The product of two fuzzy sets A and B is a new set A.B whose MF is defined as

Example

$$A = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

$$B = \{(x_1, 0.4), (x_2, 0), (x_3, 0.1)\}$$

Find A.B

Solution

$$A \cdot B = \{(x_1, 0.08), (x_2, 0), (x_3, 0.04)\}$$

Equality

Two fuzzy sets A and B are said to be equal A=B if

$$\mu_A(x) = \mu_B(x)$$

Example

$$A = \{(x_1, 0.2), (x_2, 0.8)\}$$

$$B = \{(x_1, 0.6), (x_2, 0.8)\}$$

$$C = \{(x_1, 0.2), (x_2, 0.8)\}$$

$$\boldsymbol{A \neq B}$$

$$\boldsymbol{A = C}$$

Product of a fuzzy set with a number

Multiplying a fuzzy set A by a crisp number a results in a new fuzzy set a.A with the MF :

$$\mu_{a\cdot A}(x) = a.\mu_A(x)$$

Example

$$A = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

$$a = 0.3$$

$$a.A = \{(x_1, 0.12), (x_2, 0.18), (x_3, 0.24)\}$$

Difference

Difference of two fuzzy sets A and B is a new set A-B defined as:

$$A - B = A \cap \overline{B}$$

Example

$$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}$$

$$B = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$$

Find A-B

Solution

$$\overline{B} = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

$$A - B = A \cap \overline{B} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}$$

Cardinality of a fuzzy set

For a fuzzy set $\tilde{A}$ the scalar cardinality $|\tilde{A}|$ is defined as

$$|\tilde{A}| = \sum_{x \in X} \mu_{\tilde{A}}(x)$$

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{|X|}$$ is called the relative cardinality of $\tilde{A}$.

α– Cut or level cut

- α – cut of a fuzzy set is the crisp set A that contains all the elements of the universe of discourse X whose membership grades in A are greater than or equal to the specified value α
- ie $\mu \geq \alpha$
- Strong α – cut
- Strong α – cut is a crisp set such that
- $\mu > \alpha$

Let X be the set of ages $X = \{5, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$

Example 2.3a

Consider the following fuzzy set "Young"

$$Y_{oung} = \left\{\frac{1}{5} + \frac{1}{10} + \frac{0.8}{20} + \frac{0.5}{30} + \frac{0.2}{40} + \frac{0.1}{50}\right\}$$

The level set or α-cut sets are

$$\alpha_{0.1} = [5, 10, 20, 30, 40, 50]$$

$$\alpha_{0.2} = [5, 10, 20, 30, 40]$$

$$\alpha_{0.3} = [5, 10, 20, 30] \qquad \alpha_1 = [5, 10]$$

Cardinality $|y_{oung}| = [1 + 1 + 0.8 + 0.5 + 0.2 + 0.1]$

$$= 3.6$$

Relative cardinality $= \frac{3.6}{10} = 0.36$

Properties of Fuzzy Sets

- Commutativity

$$A \cup B = B \cup A; A \cap B = B \cap A$$

- Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C; A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C); A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Idempotency

$$A \cup A = A; A \cap A = A$$

- Transitivity

$$A \subseteq B \subseteq C, \textit{ then } A \subseteq C$$

- Identity

$$A \cup \phi = A; A \cap \phi = \phi$$
$$A \cup X = X; A \cap X = A$$

- Involution

$$\overline{\overline{A}} = A$$

- DeMorgan's law

$$|\overline{A \cap B}| = \overline{A} \cup \overline{B}$$
$$|\overline{A \cup B}| = \overline{A} \cap \overline{B}$$

2.8 Algebraic operations on fuzzy sets

a. Cartesian product

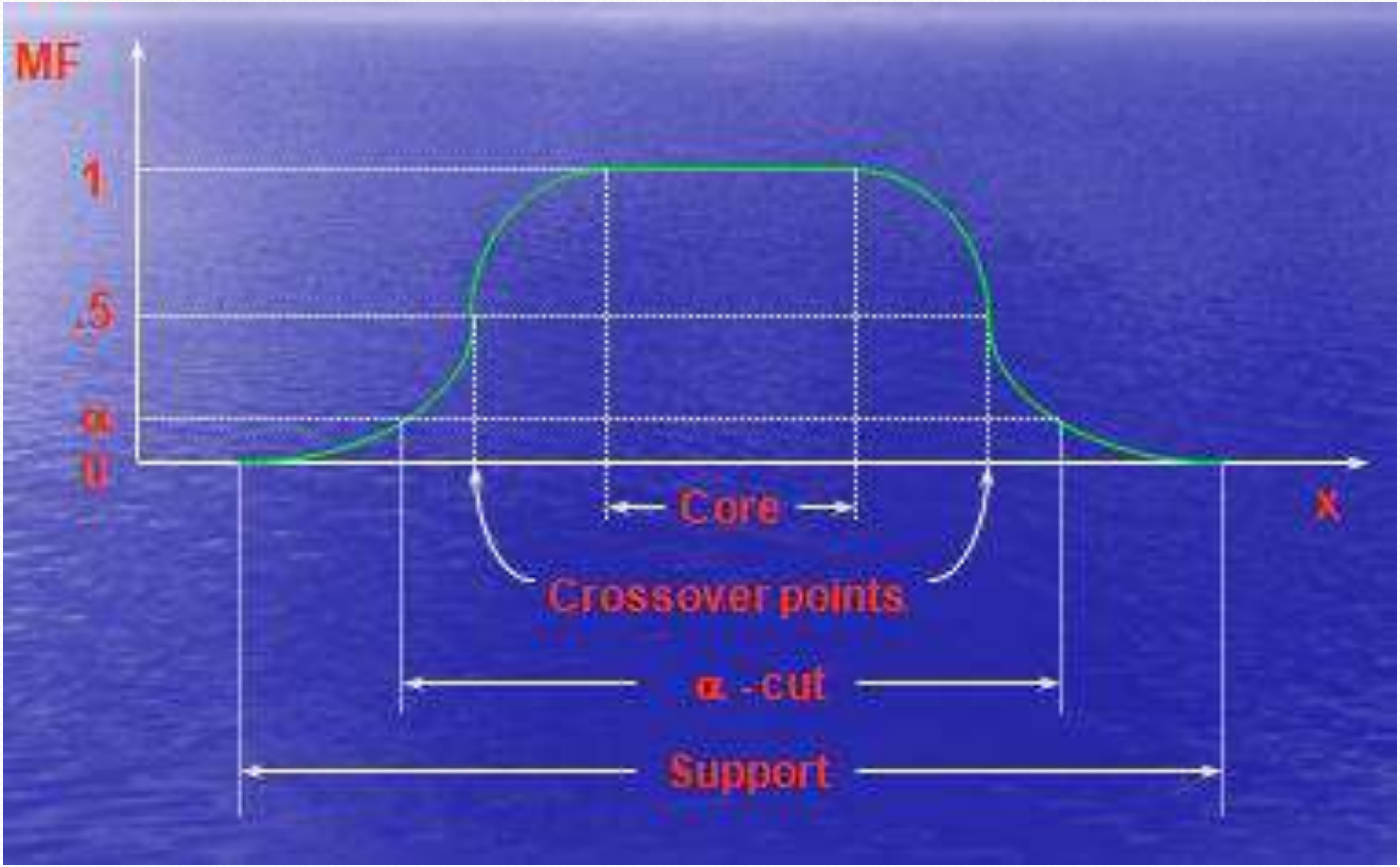
The cartesian product of two fuzzy sets A and B is a fuzzy set C denoted by A × B and defined as

$$C = A \times B = \{\mu_C(x) \mid (a, b) \mid a \in A, b \in B, \mu_C(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Let $\tilde{A} = \frac{0.4}{3} + \frac{1}{5} + \frac{0.6}{7}$ and $B = \frac{1}{5} + \frac{0.6}{7}$

Some Definitions

- Support
- Core
- Normality
- Crossover points
- Fuzzy singleton
  - α-cut, strong α -cut
- Convexity
- Bandwidth
- Symetricity


MF
1
.5
α
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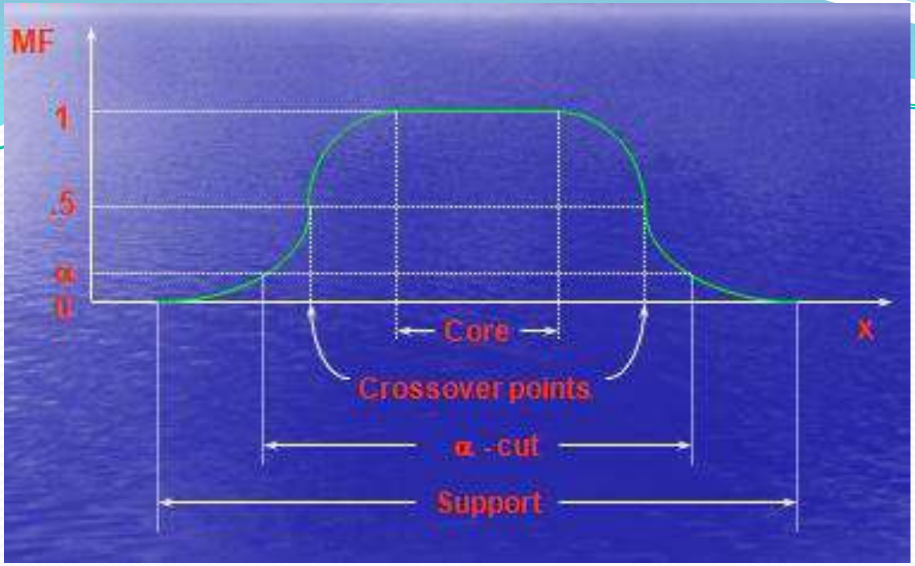
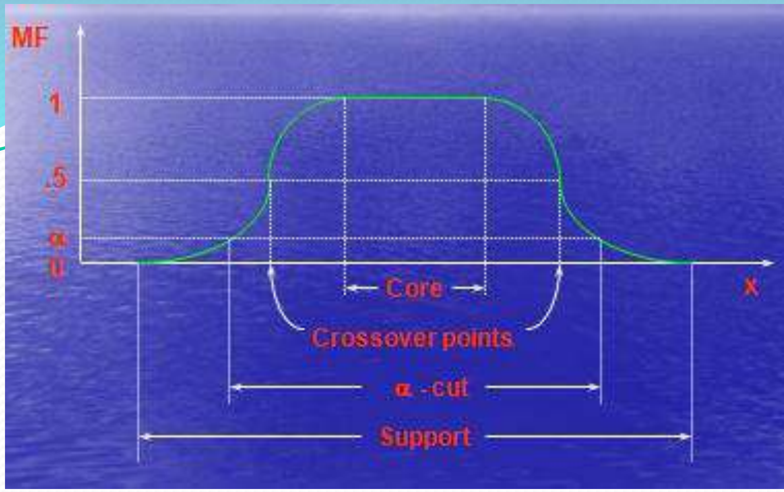
X
Core
Crossover points
α -cut
Support



- Support
- The support S(A) of a fuzzy set A is the crisp set of all the elements of the universal set (UOD) for which membership function has non-zero value

$$S(A) = \{ u \in U / \mu_A(u) > 0 \}$$

$$Support(A) = \{ x \mid \mu_A(x) > 0 \}$$



α – cut (or α level) set

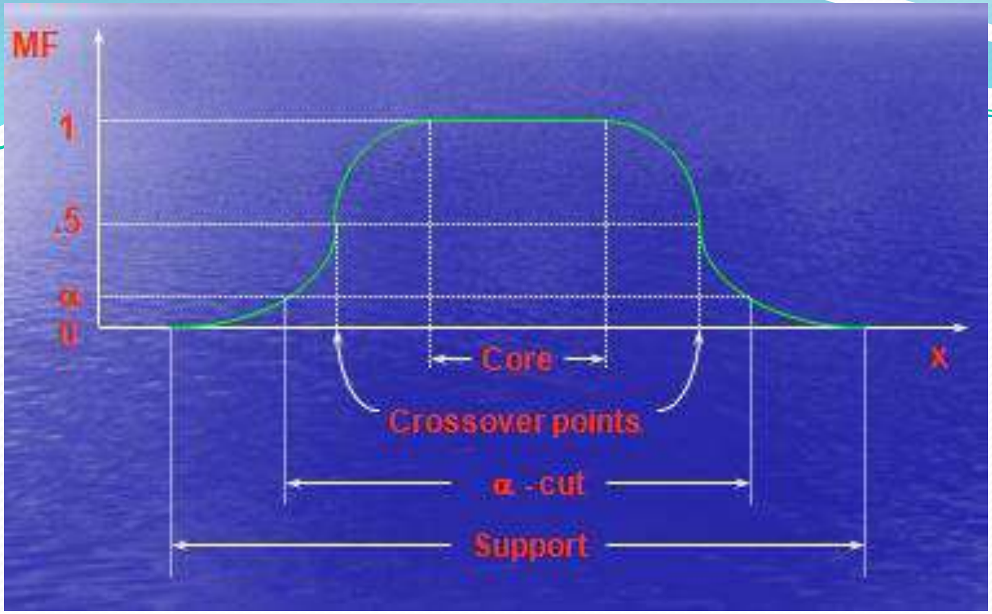
- The set of elements that belong to the fuzzy set A at least to the degree α is called the α-level-set or α-cut-set

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$$

- Strong α cut

$$A'_\alpha = \{x \mid \mu_A(x) > \alpha\}$$

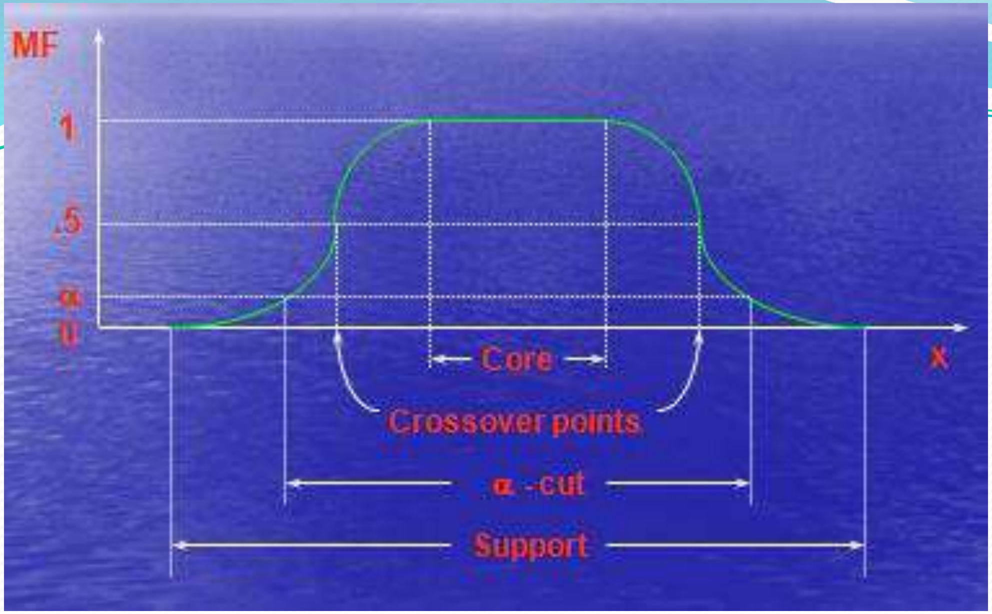
An α-cut set is crisp or fuzzy?



Crossover point

- The element of the universal set, for which the membership function has the value of 0.5, is called a crossover point.

$$Crossover(A) = \{x | \mu_A(x) = 0.5\}$$



Core:

- Is the set of all elements x in X that belong to the fuzzy set A such that $\mu_A(x)=1$:

$$Core(A) = \{x \mid \mu_A(x) = 1\}$$

Height of a fuzzy set

The height of a fuzzy set A, hgt(A) is given by a supremum of the membership function over all $u \in U$

$$hgt(A) = \sup_U \mu_A(u)$$

(Supremum in this definition means the highest possible (or almost possible) degree.)

Normality

A fuzzy set is normal if its core is nonempty. In other words, we can always find a point $x \varepsilon X$ such that $\mu_A(x)=1$

Fuzzy set operations (Recap)

- *Complement*

$$\mu_{\overline{A}} = 1 - \mu_A(x), \ for \ all \ x \in U$$

- *Algebraic sum*

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

- *Algebraic product*

$$\mu_{A.B}(x) = \mu_A(x).\mu_B(x)$$

- *Bounded sum*

$$\mu_{A \oplus B}(x) = min[1, \mu_A(x) + \mu_B(x)]$$

- *Bounded difference*

$$\mu_{A \odot B}(x) = max[0, \mu_A(x) - \mu_B(x)]$$

Properties of the fuzzy sets

- The properties of the classical set also suits for the properties of the fuzzy sets. The important properties of fuzzy set includes:
- Commutativity
- $A \cup B = B \cup A, \; A \cap B = B \cap A$
- Associativity

  $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$
- Distributivity

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

Power of a fuzzy set A^{α}:

$$\mu_A{}^{\alpha}(x) = \{\mu_A(x)\}^{\alpha}$$

If $\alpha < 1$, then it is called *dilation*

If $\alpha > 1$, then it is called *concentration*

Properties of fuzzy sets (Recap)

Idempotence :

$$A \cup A = A$$

$$A \cap A = \varnothing$$

$$A \cup \varnothing = A$$

$$A \cap \varnothing = \varnothing$$

Transitivity :

If $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$

Involution :

$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

(5) *Consider 2 given fuzzy sets,*

$$A = \{\frac{0.2}{1} + \frac{0.3}{2} + \frac{0.4}{3} + \frac{0.5}{4}\}$$
$$B = \{\frac{0.1}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{0.1}{4}\}$$

Find,

(*a*) *algebraic sum*

(*b*) *algebraic product*

(*c*) *bounded sum*

(*d*) *bounded difference*

(a) *Algebraic sum*

$$\mu_{A+B}(X) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$$

$$= \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.5}{4}\} - \{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\}$$

$$= \{\frac{0.28}{1} + \frac{0.44}{2} + \frac{0.52}{3} + \frac{1}{4}\}$$

(b) *Algebraic product*

$$\mu_{A \cdot B}(X) = \mu_A(x) \cdot \mu_B(x)$$

$$= \{\frac{0.02}{1} + \frac{0.06}{2} + \frac{0.08}{3} + \frac{0.5}{4}\}$$

(c) Bounded sum

$$\mu_{A \bigoplus B}(X) = min[1, \mu_A(x) + \mu_B(x)]$$
$$= min\{1, \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1.5}{4}\}\}$$
$$= \{\frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{1}{4}\}$$

(d) Bounded difference

$$\mu_{A \ominus B}(X) = max[0, \mu_A(x) - \mu_B(x)]$$
$$= max\{0, \{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\}\}$$
$$= \{\frac{0.1}{1} + \frac{0.1}{2} + \frac{0.2}{3} + \frac{0.5}{4}\}$$

Fuzzy Relations

Cartesian Product

- The **Cartesian product** of two **sets** A and B, denoted $A \times B$, is the **set** of all possible ordered pairs where the elements of A are first and the elements of B are second.
- In **set**-builder notation,
- $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Eg:

- Consider,

$$X = \{p, q, r\}$$
$$Y = \{2, 4, 6\}$$

Cartesian product of these two sets, $X \times Y$, is,

$\{(p, 2), (p, 4), (p, 6), (q, 2), (q, 4), (q, 6), (r, 2), (r, 4), (r, 6)\}$

- •The elements in two sets A and B are given as
- $A = \{0, 1\}$ and $B = \{a, b, c\}$.
- Various Cartesian products of these two sets can be written as shown:•
- $A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$
- $B \times A = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$
- $A \times A = A2= \{(0, 0), (0, 1), (1, 0), (1, 1)\}$
- $B \times B = B2= \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a),$
- $(c, b), (c, c)\}$
- Note that $A \times B \neq B \times A$ i.e cartesian product is not commutative

Classical Relation/ Crisp Relation

- A subset of the cartesian product $A_1 \times A_2 \ldots \ldots X A_r$ is called an r-ary relation over $A_1, A_2, \ldots \ldots, A_r$.

The most common case is when r=2 the relation is subset of cartesian product $A_1 \times A_2$.(binary relation)

This means a binary relation from A_1 into A_2.

If three, four, five sets are involved in the subset, then we call them ternary, quaternary and quinary relations.

Classical Relations

- An *r ary* relation over $A_1; A_2; ::::; A_r$ is a *subset* of the Cartesian product $A_1 A_2 :::: A_r$.

r	r ary relation
2	binary
3	ternary
4	quaternary
5	quinary

- Consider,

$$X = \{p, q, r\}$$
$$Y = \{2, 4, 6\}$$

Cartesian product of these two sets, $X \times Y$, is,
$\{(p,2),(p,4),(p,6),(q,2),(q,4),(q,6),(r,2),(r,4),(r,6)\}$
From this set one may select a subset such that,

$$R = \{(p,2),(q,4),(r,4),(r,6)\}$$

Relation matrix is,

$$\begin{array}{cc} & \begin{array}{ccc} 2 & 4 & 6 \end{array} \\ \begin{array}{c} p \\ q \\ r \end{array} & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \end{array}$$

Different ways t to represent a relation

- Matrix form
- Coordinate Diagram
- Mapping of the Relation

Matrix form

Consider,

$$X = \{p, q, r\}$$
$$Y = \{2, 4, 6\}$$

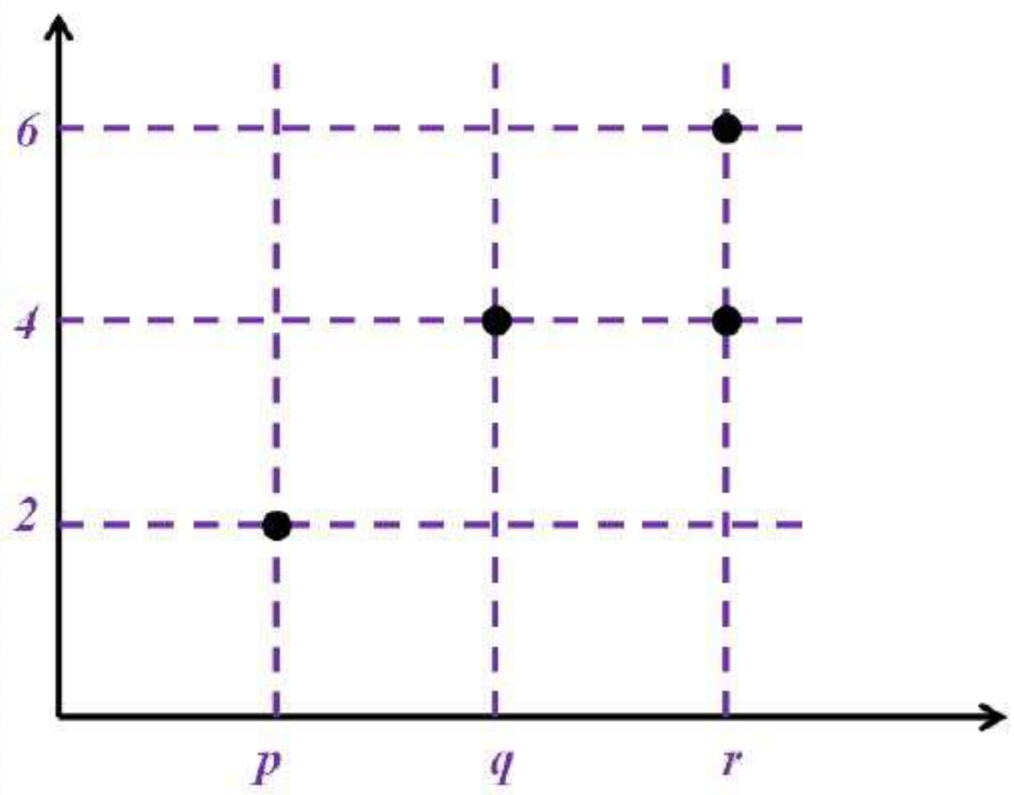
$$R = \{(p,2),(q,4),(r,4),(r,6)\}$$

Relation matrix is,

$$\begin{array}{c} \\ p \\ q \\ r \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

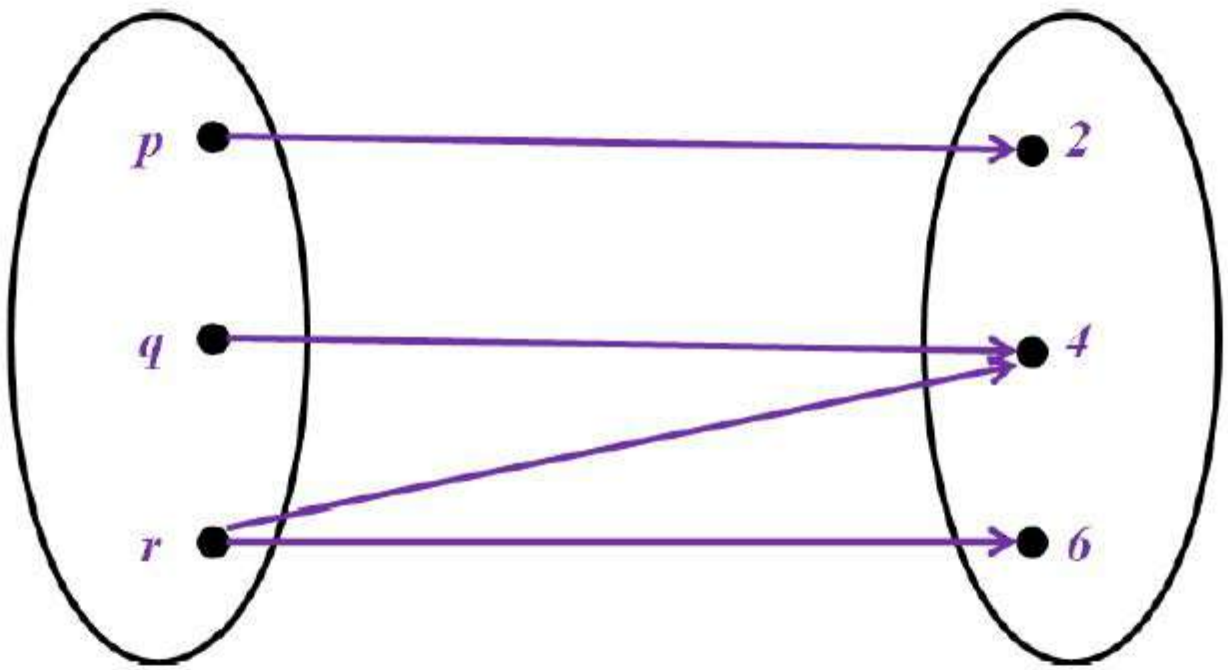
Coordinate diagram of the relation

$$R = \{(p, 2), (q, 4), (r, 4), (r, 6)\}$$



Mapping of the relation

$$R = \{(p, 2), (q, 4), (r, 4), (r, 6)\}$$



Cardinality of classical relation

When the cardinality of,

$$X = n_X \text{ and}$$
$$Y = n_Y;$$

then the cardinality of relation R between the two universe is,

$$n_{X \times Y} = n_X \times n_Y$$

The cardinality of the power set is given by,

$$n_{P(X \times Y)} = 2^{(n_X \times n_Y)}$$

Operations on Classical Relations

- Union

$$R \cup S \longrightarrow \chi_{R \cup S}(x, y); \chi_{R \cup S}(x, y) = max[\chi_R(x, y), \chi_S(x, y)]$$

- Intersection

$$R \cap S \longrightarrow \chi_{R \cap S}(x, y); \chi_{R \cap S}(x, y) = min[\chi_R(x, y), \chi_S(x, y)]$$

- Complement

$$\overline{R} \longrightarrow \chi_{\overline{R}}(x, y); \chi_{\overline{R}}(x, y) = 1 - \chi_R(x, y)$$

- Containment

$$R \subset S \longrightarrow \chi_R(x, y); \chi_R(x, y) \leq \chi_S(x, y)$$

- Identity

$$\phi \longrightarrow \phi_R \ and \ X \longrightarrow E_R$$

Properties of crisp relations

- **Commutativity**
- **Associativity**
- **Distributivity**
- **Involution**
- **Idempotency**
- **Excluded middle laws**
- **DeMorgan' s law**

Composition of Classical Relations

- The operation executed on two compatible binary relations to get a single binary relation is called *composition*.
- Let R be a relation that maps elements from X to Y and S be a relation that maps elements from Y to Z. R and S are compatible if,

$$R \subseteq X \times Y \text{ and } S \subseteq Y \times Z$$

- The composition between the two relations is denoted by $R \circ S$.

Example

- Consider the universal sets,

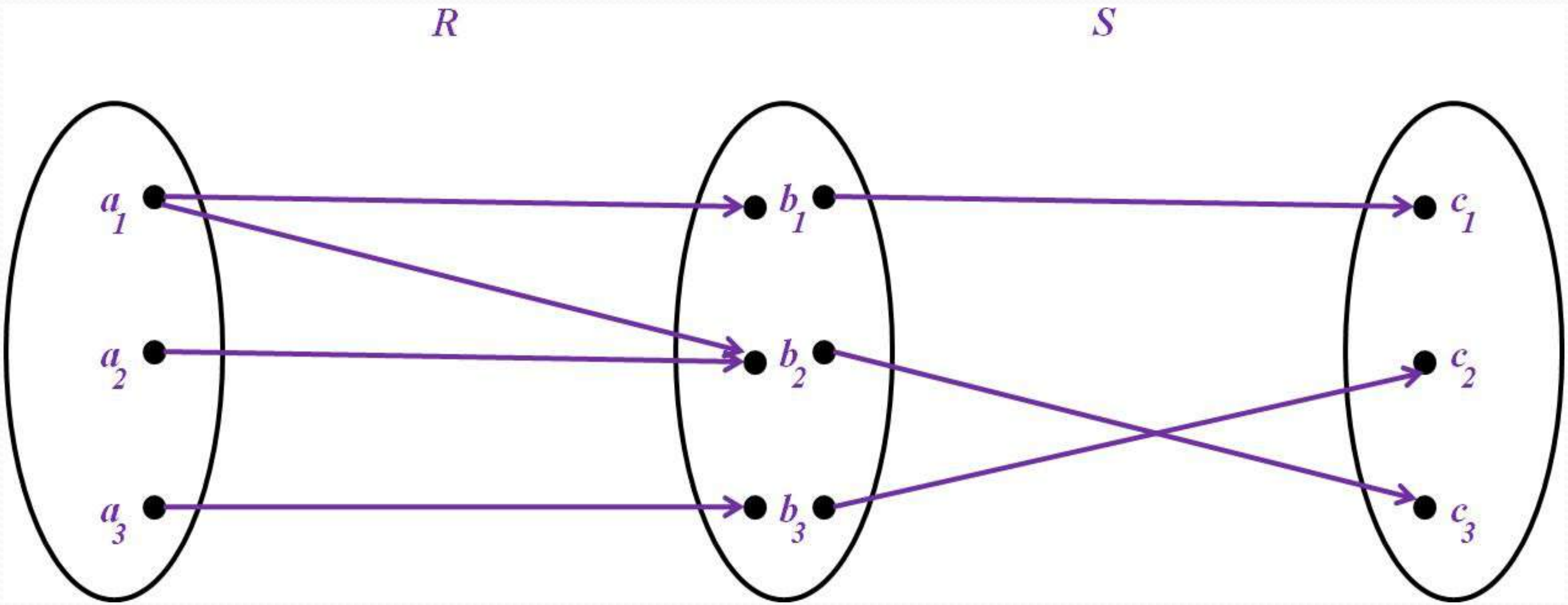
$$X = \{a_1, a_2, a_3\}$$
$$Y = \{b_1, b_2, b_3\}$$
$$Z = \{c_1, c_2, c_3\}$$

Let the relations R and S be formed as,

$$R = X \times Y = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_3, b_3)\}$$
$$S = Y \times Z = \{(b_1, c_1), (b_2, c_3), (b_3, c_2)\}$$

$$T = R \circ S = \{(a_1, c_1), (a_2, c_3), (a_3, c_2), (a_1, c_3)\}$$

Illustration of relation R and S



The composition operations are of two types:

1. Max min composition

2. Max product composition

Max-min Composition

- The max–min composition is defined by the function theoretic expression as:

$$T = R \circ S$$

$$\chi_T(x, z) = \vee\{\chi_R(x, y) \wedge \chi_S(y, z)\}$$

Max-product Composition

- The max−product composition is defined by the function theoretic expression as:

$$T = R \circ S$$

$$\chi_T(x, z) = \vee\{\chi_R(x, y) \cdot \chi_S(y, z)\}$$

Example:

Given

$X = \{1, 3, 5\}; Y = \{1, 3, 5\}; R = \{(x, y)|y = x + 2\}; S = \{(x, y)|x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have

$R = \{(1, 3), (3, 5)\}$

$S = \{(1, 3), (1, 5), (3, 5)\}$

$$R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and } S = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition $R \circ S =$

$$\begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of Composition Operation

Associative	$(R \circ S) \circ M = R \circ (S \circ M)$
Commutative	$R \circ S = S \circ R$
Inverse	$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

Fuzzy Relations

- *Fuzzy relations* relate elements of one universe to those of another universe through the Cartesian product of the two universes.
- Based on the concept that everything is related to some extent or unrelated.
- A fuzzy relation is a fuzzy set defined on the Cartesian product of classical sets,

$$\{X_1, X_2, ..., X_n\}$$

where tuples,

$$(x_1, x_2, ..., x_n)$$

may have varying degrees of membership,

$$\mu_R(x_1, x_2, ..., x_n)$$

within the relation.

$$R(X_1, X_2, ..., X_n) = \int_{X_1 \times X_2 \times .... \times X_n} \frac{\mu_R(x_1, x_2, ..., x_n)}{(x_1, x_2, ..., x_n)}, x_i \in X_i$$

Representation of Fuzzy Relations

- Fuzzy Matrix
- Simple Fuzzy Graph
- Bipartite Graph

Fuzzy Matrix

- Let,

$$X = \{x_1, x_2, ..., x_n\} \text{ and } Y = \{y_1, y_2, \cdots, y_n\}$$

Fuzzy relation $R(x, y)$ can be expressed as an $n \times m$ matrix as:

$$R(x,y) = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & ..... & \mu_R(x_1, y_m) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & ..... & \mu_R(x_2, y_m) \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & ..... & \mu_R(x_n, y_m) \end{bmatrix}$$

- The matrix representing a fuzzy relation is called *Fuzzy matrix*.

Example:

X = { typhoid, viral, cold } and Y = { running nose, high temp, shivering }

The fuzzy relation R is defined as

$$
\begin{array}{c} \\ typhoid \\ viral \\ cold \end{array}
\begin{array}{c}
\begin{array}{ccc} runningnose & hightemperature & shivering \end{array} \\
\left[\begin{array}{ccc} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{array} \right]
\end{array}
$$

Fuzzy Relations Example

- Let,

$$X = \{x_1, x_2, x_3, x_4\}\ and\ Y = \{y_1, y_2, y_3, y_4\}$$

Let R be a relation from X and Y given by,

$$R = \frac{0.2}{(x_1, y_3)} + \frac{0.4}{(x_1, y_2)} + \frac{0.1}{(x_2, y_2)} + \frac{0.6}{(x_2, y_3)} + \frac{1.0}{(x_3, y_3)} + \frac{0.5}{(x_3, y_1)}$$

Fuzzy matrix for relation R is,

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0.1 & 0.6 \\ 0.5 & 0 & 1.0 \end{bmatrix} \end{array}$$

Fuzzy graph

- *Fuzzy graph* is a graphical representation of binary fuzzy relation.
- Each element in X and Y corresponds to a node in the fuzzy graph.
- The connection links are established between the nodes by the elements of $X \times Y$ with nonzero membership grades in $R(X, Y)$.
- The links may also be present in the form of arcs.
- Links are labeled with the membership values as $\mu_R(x_i, y_j)$.

Example

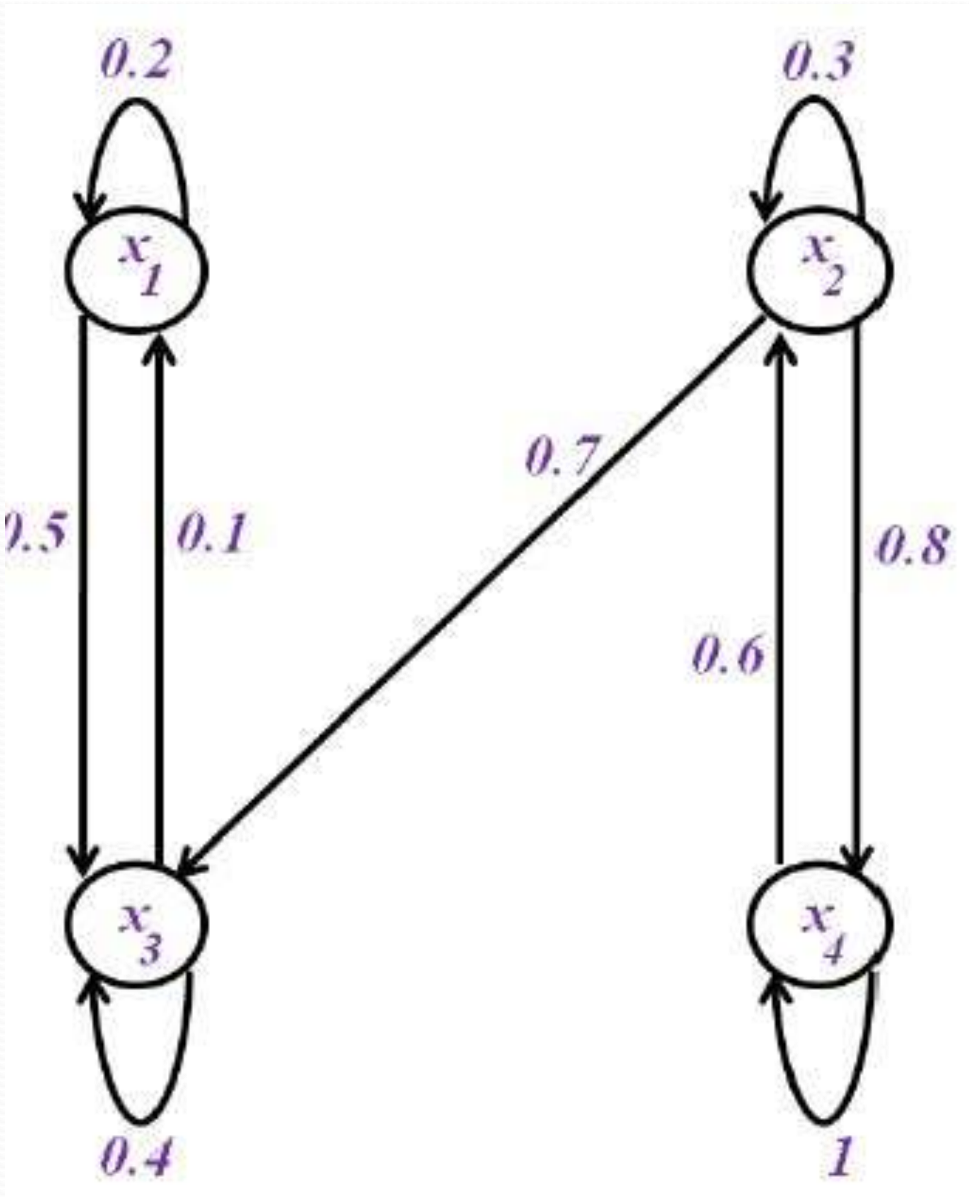
- Consider,

$$X = \{x_1, x_2, x_3, x_4\}$$

Binary fuzzy relation on X as,

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 1 \end{bmatrix} \end{array}$$

Fuzzy Graph



Bipartite Graph

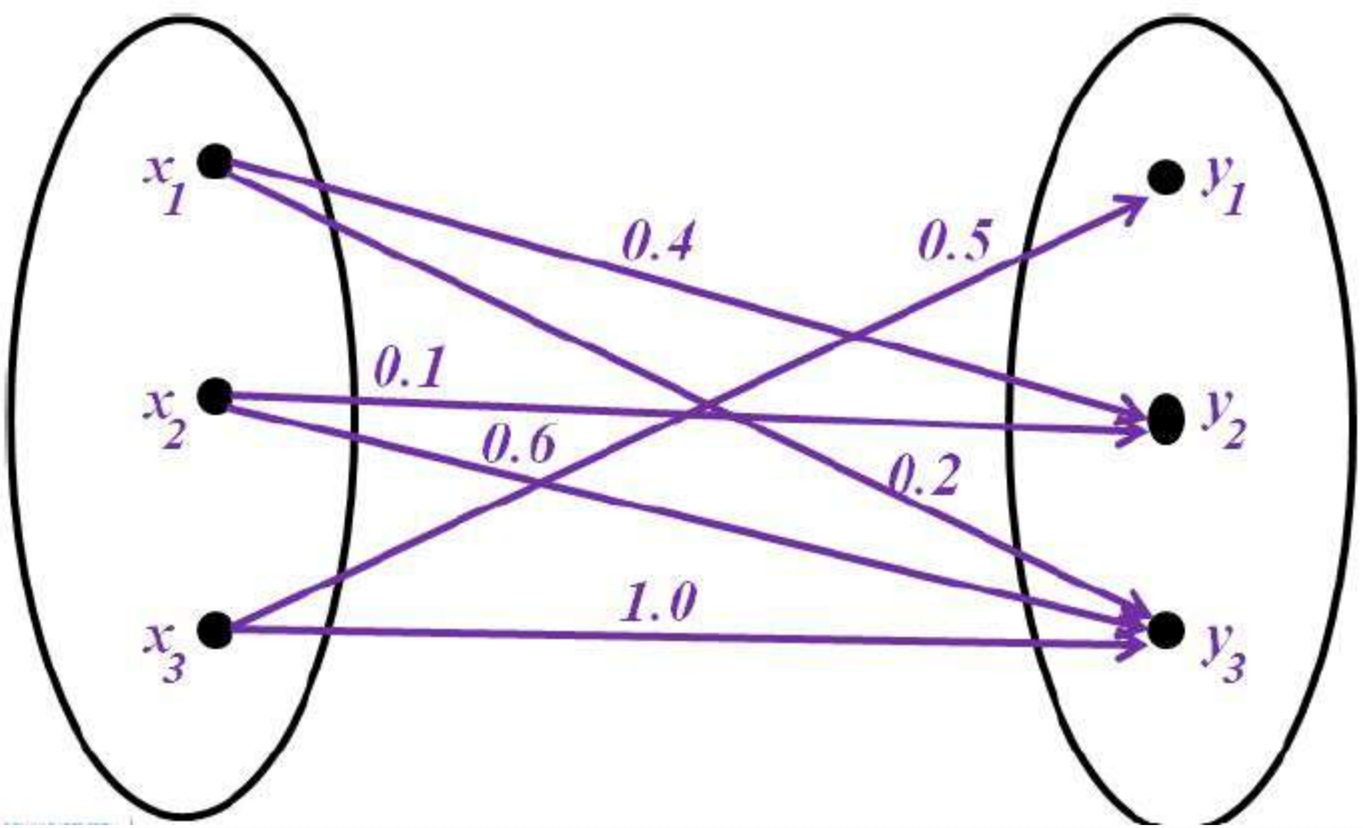
- When $x \neq y$, the link connecting the two nodes is an undirected binary graph called *bipartite graph*.
- Each of the sets X and Y can be represented by a set of nodes such that nodes corresponding to one set are clearly differentiated from the nodes representing the other set.

Example

Fuzzy matrix for relation R is,

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0.2 \\ 0 & 0.1 & 0.6 \\ 0.5 & 0 & 1.0 \end{bmatrix} \end{array}$$

Bipartite Graph



Fuzzy Relations

A fuzzy relation R is a 2D MF:

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \mid (x, y) \in X \times Y \right\}$$

Fuzzy Cartesian Product

Suppose

A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operation on Fuzzy Relations

- Union

$$\mu_{R \cup S}(x, y) = max\{\mu_R(x, y), \mu_S(x, y)\}$$

- Intersection

$$\mu_{R \cap S}(x, y) = min\{\mu_R(x, y), \mu_S(x, y)\}$$

- Complement

$$\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y)$$

- Containment

$$R \subset S \Longrightarrow \mu_R(x, y) \leq \mu_S(x, y)$$

- Inverse

$$R^{-1}(y, x) = R(x, y) \text{ for all pairs } (y, x) \in Y \times X$$

- Projection

$$\mu_{[R \downarrow Y]}(x, y) = max_x \mu_R(x, y)$$

Properties of Fuzzy Relations

- Commutativity
- Associativity
- Distributivity
- Identity
- Idempotency
- DeMorgan's law

FUZZY COMPOSITION

Fuzzy Compositions

- Let A be a fuzzy set on universe X and B be a fuzzy set on universe Y.
- the cartesian product over A and B results in fuzzy relation $R.ie,$

$$A \times B = R$$

where

$$R \subset X \times Y$$

- The membership function is given by,

$$\mu_R(x, y) = \mu_{A \times B}(x, y) = min[\mu_A(x), \mu_B(y)]$$

Fuzzy Composition Techniques

- Max-min Composition
- Max-product Composition
- Min-max Composition

- *Max–min composition*
  - Let R be fuzzy relation on cartesian space $X \times Y$ and S be fuzzy relation on cartesian space $Y \times Z$.
  - Max–min composition of $R(X, Y)$ and $S(Y, Z)$,

$$\mu_T(x,z) = \mu_{R \circ S}(x,z)$$
$$= max_{y \in Y}\{min[\mu_R(x,y), \mu_S(y,z)]\}$$
$$= \vee_{y \in Y}[\mu_R(x,y) \wedge \mu_S(y,z)] \forall x \in X, z \in Z$$

Example:

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2)$; $Z = (z_1, z_2, z_3)$;

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$\mu_{R \circ S}(x_1, y_1) = max\{min(x_1, y_1), min(y_1, z_1), min(x_1, y_2), min(y_2, z_1)\}$
$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5$ and so on.

- *Max–product composition*
  - Let R be fuzzy relation on cartesian space $X \times Y$ and S be fuzzy relation on cartesian space $Y \times Z$.
  - Max–min composition of $R(X, Y)$ and $S(Y, Z)$,

$$\begin{aligned} \mu_T(x,z) &= \mu_{R \circ S}(x,z) \\ &= max_{y \in Y}\{\mu_R(x,y) \cdot \mu_S(y,z)\} \\ &= \vee_{y \in Y}[\mu_R(x,y) \cdot \mu_S(y,z)] \forall x \in X, z \in Z \end{aligned}$$

Consider fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc} 0.7 & 0.6 \\ 0.8 & 0.3 \end{array}\right] \end{array}, \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_2 \end{array} \\ \left[\begin{array}{ccc} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{array}\right] \end{array}.$$

Find the relation $T = \underset{\sim}{R} \circ \underset{\sim}{S}$ using max–min and max–product composition.

Solution. **Max–Min Composition**

$$
\begin{aligned}
T &= \underset{\sim}{R} \circ \underset{\sim}{S} \\
\mu_T(x_1, z_1) &= \max[\min(0.7, 0.8), \min(0.6, 0.1)] \\
&= \max[0.7, 0.1] \\
&= 0.7, \\
\mu_T(x_1, z_2) &= \max[\min(0.7, 0.5), \min(0.6, 0.6)] \\
&= \max[0.5, 0.6] \\
&= 0.6, \\
\mu_T(x_1, z_3) &= \max[\min(0.7, 0.4), \min(0.6, 0.7)] \\
&= \max[0.4, 0.6] \\
&= 0.6 \\
\mu_T(x_2, z_1) &= \max[\min(0.8, 0.8), \min(0.3, 0.1)] \\
&= \max[0.8, 0.1] \\
&= 0.8, \\
\mu_T(x_2, z_2) &= \max[\min(0.8, 0.5), \min(0.3, 0.6)] \\
&= \max[0.5, 0.3] \\
&= 0.5, \\
\mu_T(x_2, z_3) &= \max[\min(0.8, 0.4), \min(0.3, 0.7)] \\
&= 0.4,
\end{aligned}
$$

$$
\underset{\sim}{S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.6 \\ 0.8 & 0.5 & 0.4 \end{bmatrix} \end{array} .
$$

Max–Product Composition

$$\mu_T(x_1, z_1) = \max[\min(0.7 \times 0.8), \min(0.6 \times 0.1)]$$
$$= \max[0.56, 0.06]$$
$$= 0.56,$$
$$\mu_T(x_1, z_2) = \max[\min(0.7 \times 0.5), \min(0.6 \times 0.6)]$$
$$= \max[0.35, 0.36]$$
$$= 0.36,$$
$$\mu_T(x_1, z_3) = \max[\min(0.7 \times 0.4), \min(0.6 \times 0.7)]$$
$$= \max[0.28, 0.42]$$
$$= 0.42$$
$$\mu_T(x_2, z_1) = \max[\min(0.8 \times 0.8), \min(0.3 \times 0.1)]$$
$$= \max[0.64, 0.03]$$
$$= 0.64,$$
$$\mu_T(x_2, z_2) = \max[\min(0.8 \times 0.5), \min(0.3 \times 0.6)]$$
$$= \max[0.40, 0.18]$$
$$= 0.40,$$
$$\mu_T(x_2, z_3) = \max[\min(0.8 \times 0.4), \min(0.3 \times 0.7)]$$
$$= \max[0.32, 0.21]$$
$$= 0.32,$$
$$\underset{\sim}{T} = \begin{bmatrix} 0.56 & 0.36 & 0.35 \\ 0.64 & 0.40 & 0.32 \end{bmatrix}.$$

- *Min–max composition*
  - Let R be fuzzy relation on cartesian space $X \times Y$ and S be fuzzy relation on cartesian space $Y \times Z$.
  - Max–min composition of $R(X, Y)$ and $S(Y, Z)$,

$$\mu_T(x, z) = \mu_{R \circ S}(x, z)$$
$$= min_{y \in Y}\{max[\mu_R(x, y), \mu_S(y, z)]\}$$
$$= \wedge_{y \in Y}[\mu_R(x, y) \vee \mu_S(y, z)] \forall x \in X, z \in Z$$

Properties of Fuzzy Composition

1. $R \circ S = S \circ R$
2. $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$
3. $(R \circ S) \circ M = R \circ (S \circ M)$

Problems

(1) *Consider the following two fuzzy sets:*

$$A = \{\frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{1}{x_3}\} \text{ and}$$
$$B = \{\frac{0.4}{y_1} + \frac{0.9}{y_2}\}$$

Perform the cartesian product over these given fuzzy sets.

$$\mu_R(x_1, y_1) = min[\mu_A(x_1), \mu_B(y_1)] = min[0.3, 0.4] = 0.3$$
$$\mu_R(x_1, y_2) = min[\mu_A(x_1), \mu_B(y_2)] = min[0.3, 0.9] = 0.3$$
$$\mu_R(x_2, y_1) = min[\mu_A(x_2), \mu_B(y_1)] = min[0.7, 0.4] = 0.4$$
$$\mu_R(x_2, y_2) = min[\mu_A(x_2), \mu_B(y_2)] = min[0.7, 0.9] = 0.7$$
$$\mu_R(x_3, y_1) = min[\mu_A(x_3), \mu_B(y_1)] = min[1, 0.4] = 0.4$$
$$\mu_R(x_3, y_2) = min[\mu_A(x_3), \mu_B(y_2)] = min[1, 0.9] = 0.9$$

$$\mu_R(x_1, y_1) = min[\mu_A(x_1), \mu_B(y_1)] = min[0.3, 0.4] = 0.3$$
$$\mu_R(x_1, y_2) = min[\mu_A(x_1), \mu_B(y_2)] = min[0.3, 0.9] = 0.3$$
$$\mu_R(x_2, y_1) = min[\mu_A(x_2), \mu_B(y_1)] = min[0.7, 0.4] = 0.4$$
$$\mu_R(x_2, y_2) = min[\mu_A(x_2), \mu_B(y_2)] = min[0.7, 0.9] = 0.7$$
$$\mu_R(x_3, y_1) = min[\mu_A(x_3), \mu_B(y_1)] = min[1, 0.4] = 0.4$$
$$\mu_R(x_3, y_2) = min[\mu_A(x_3), \mu_B(y_2)] = min[1, 0.9] = 0.9$$

$$R = A \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.7 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

(2) *Two fuzzy relations are given by,*

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array} \; and$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Obtain fuzzy relation T as a composition between the fuzzy relations R and S.

(a) Max–min composition

$$\begin{aligned}
\mu_T(x_1, z_1) &= max\{min[\mu_R(x_1, y_1), \mu_S(y_1, z_1)], \\
&\qquad min[\mu_R(x_1, y_2), \mu_S(y_2, z_1)]\} \\
&= max\{min[0.6, 1], min[0.3, 0.8]\} \\
&= max\{0.6, 0.3\} = 0.6 \\
\mu_T(x_1, z_2) &= max\{min[\mu_R(x_1, y_1), \mu_S(y_1, z_2)], \\
&\qquad min[\mu_R(x_1, y_2), \mu_S(y_2, z_2)]\} \\
&= max\{min[0.6, 0.5], min[0.3, 0.4]\} \\
&= max\{0.5, 0.3\} = 0.5 \\
\mu_T(x_1, z_3) &= max\{min[\mu_R(x_1, y_1), \mu_S(y_1, z_3)], \\
&\qquad min[\mu_R(x_1, y_2), \mu_S(y_2, z_3)]\} \\
&= max\{min[0.6, 0.3], min[0.3, 0.7]\} \\
&= max\{0.3, 0.3\} = 0.3
\end{aligned}$$

$$\begin{aligned}\mu_T(x_2, z_1) &= max\{min[\mu_R(x_2, y_1), \mu_S(y_1, z_1)],\\ &\qquad min[\mu_R(x_2, y_2), \mu_S(y_2, z_1)]\}\\ &= max\{min[0.2, 1], min[0.9, 0.8]\}\\ &= max\{0.2, 0.8\} = 0.8\end{aligned}$$

$$\begin{aligned}\mu_T(x_2, z_2) &= max\{min[\mu_R(x_2, y_1), \mu_S(y_1, z_2)],\\ &\qquad min[\mu_R(x_2, y_2), \mu_S(y_2, z_2)]\}\\ &= max\{min[0.2, 0.5], min[0.9, 0.4]\}\\ &= max\{0.2, 0.4\} = 0.4\end{aligned}$$

$$\begin{aligned}\mu_T(x_2, z_3) &= max\{min[\mu_R(x_2, y_1), \mu_S(y_1, z_3)],\\ &\qquad min[\mu_R(x_2, y_2), \mu_S(y_2, z_3)]\}\\ &= max\{min[0.2, 0.3], min[0.9, 0.7]\}\\ &= max\{0.2, 0.7\} = 0.7\end{aligned}$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.9 \end{bmatrix} \end{array} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

$$T = R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Real life Example-1

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants
$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, then R can be stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

Also, consider T be the another relation on $D \times S$, which is given by

$$
\mathbf{T} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}
$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

$$\mathbf{T} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Hint: Find $R \circ T$, and verify that

$$R \circ \mathbf{T} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Real life Example-2

(3) *For a speed control of DC motor, the membership functions of series resistance, armature current and speed are given as follows:*

$$SR = \{\frac{0.4}{30} + \frac{0.6}{60} + \frac{1.0}{100} + \frac{0.1}{120}\}$$
$$I = \{\frac{0.2}{20} + \frac{0.3}{40} + \frac{0.6}{60} + \frac{0.8}{80} + \frac{1.0}{100} + \frac{0.2}{120}\}$$
$$N = \{\frac{0.35}{500} + \frac{0.67}{1000} + \frac{0.97}{1500} + \frac{0.25}{1800}\}$$

Compute relation T for relating series resistance to motor speed. Perform max–min composition only.

$$R = SR \times I = \begin{array}{c} \\ 30 \\ 60 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccccc} 20 & 40 & 60 & 80 & 100 & 120 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.4 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.6 & 0.6 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.6 & 0.8 & 1.0 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

$$S = I \times N = \begin{array}{c} \\ 20 \\ 40 \\ 60 \\ 80 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccc} 500 & 1000 & 1500 & 1800 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.25 \\ 0.35 & 0.6 & 0.6 & 0.25 \\ 0.35 & 0.67 & 0.8 & 0.25 \\ 0.35 & 0.67 & 0.97 & 0.25 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

$$T = R \circ S = \begin{array}{c} \\ 30 \\ 60 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccc} 500 & 1000 & 1500 & 1800 \end{array} \\ \begin{bmatrix} 0.35 & 0.4 & 0.4 & 0.25 \\ 0.35 & 0.6 & 0.6 & 0.25 \\ 0.35 & 0.67 & 0.97 & 0.25 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

Tolerance and Equivalence Relations

- Relations possesses various useful properties
- Three characteristic properties are :
- Reflexivity
- Symmetry
- Transitivity

- A relation is said to be reflective if every vertex (node) in the graph originates a single loop as shown

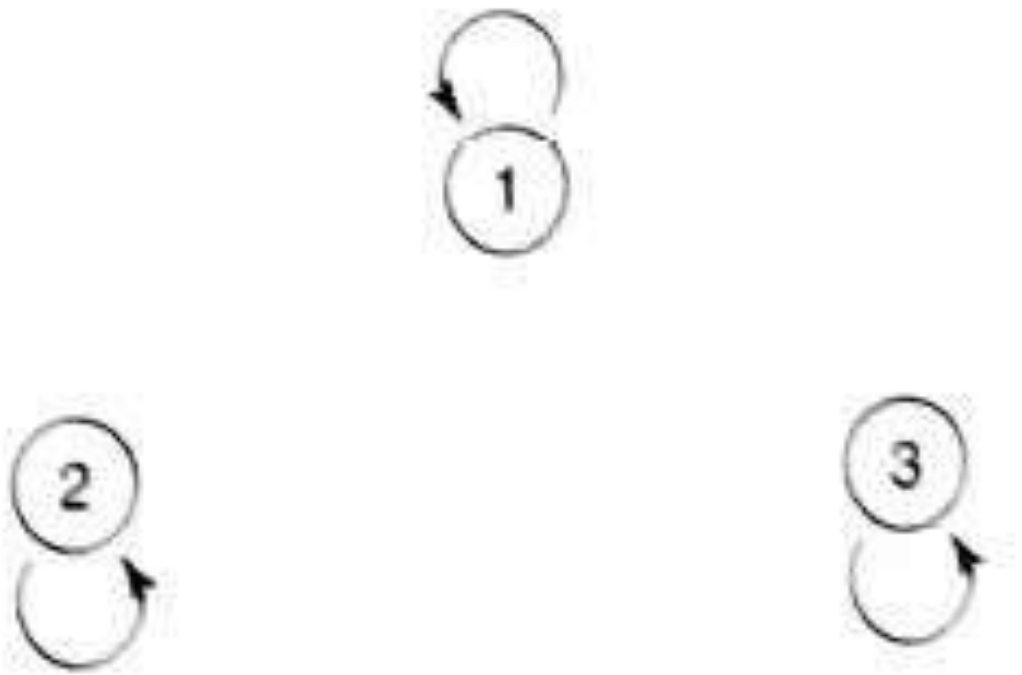


Figure Three-vertex node – reflexive property.

- A relation is said to be symmetric if every edge pointing from vertex i to vertex j , there is an edge pointing in the opposite direction i.e. from vertex j to vertex i, where i,j =1,2,3…. Figure represents a symmetric relation

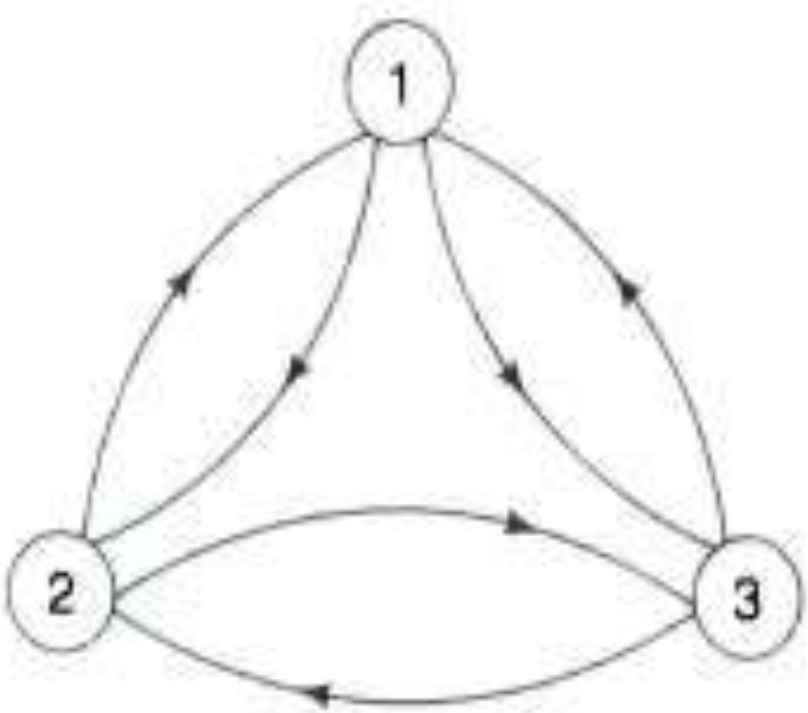


Figure Three-vertex node – symmetry property.

- A relation is said to be transitive if for every pair of edges in the graph- one pointing from vertex i to vertex j and other pointing from vertex j to vertex k , there is an edge pointing from vertex i to vertex k

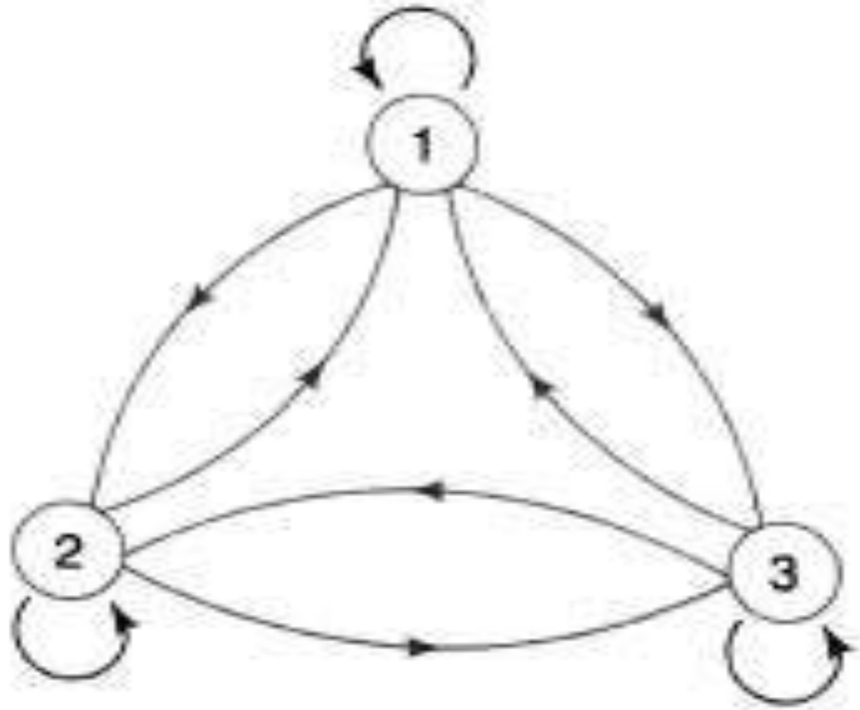


Figure Three-vertex graph – transitive property.

Classical Equivalence Relation

Let relation R on universe X be a relation from X to X. Relation R is an equivalence relation if the three properties are satisfied:

1. Reflexivity
2. Symmetry
3. Transitivity

The function theoretic forms of representation of these properties are as follows:

1. Reflexivity

$$\chi_R(x_i, x_i) = 1 \text{ or } (x_i, x_i) \in R$$

2. Symmetry

$$\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$$
$$\text{i.e., } (x_i, x_j) \in R \Rightarrow (x_j, x_i) \in R$$

3. Transitivity

$$\chi_R(x_i, x_j) \text{ and } \chi_R(x_j, x_k) = 1, \text{ so } \chi_R(x_i, x_k) = 1$$

$$\text{i.e., } (x_i, x_j) \in R(x_j, x_k) \in R, \text{ so } (x_i, x_k) \in R$$

The best example of an equivalence relation is the relation of similarity among triangles.

Classical Tolerance Relation

A tolerance relation R_1 on universe X is one where the only the properties of reflexivity and symmetry are satisfied. The tolerance relation can also be called proximity relation. An equivalence relation can be formed from tolerance relation R_1 by $(n-1)$ compositions within itself, where n is the cardinality of the set that defines R_1, here it is X, i.e.

$$\underbrace{R_1^{n-1}}_{\substack{\text{Tolerance}\\ \text{relation}}} = R_1 \circ R_1 \circ \cdots \circ R_1 = \underbrace{R}_{\substack{\text{Equivalence}\\ \text{relation}}}$$

Fuzzy Equivalence Relation

Let $\underset{\sim}{R}$ be a fuzzy relation on universe X, which maps elements from X to X. Relation $\underset{\sim}{R}$ will be a fuzzy equivalence relation if all the three properties – reflexive, symmetry and transitivity – are satisfied. The membership function theoretic forms for these properties are represented as follows:

1. Reflexivity

$$\mu_{\underset{\sim}{R}}(x_i, x_i) = 1 \quad \forall x \in X$$

If this is not the case for few $x \in X$, then $R(X, X)$ is said to be irreflexive.

2. Symmetry

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i) \text{ for all } x_i, x_j \in X$$

If this is not satisfied for few $x_i, x_j \in X$, then $R(X, X)$ is called asymmetric.

3. Transitivity

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2$$
$$\Rightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda$$

where

$$\lambda = \min[\lambda_1, \lambda_2]$$
$$\text{i.e., } \mu_{\underset{\sim}{R}}(x_i, x_j) \geq \max_{x_j \in X} \min[\mu_{\underset{\sim}{R}}(x_i, x_j), \mu_{\underset{\sim}{R}}(x_j, x_k)] \quad \forall (x_i, x_k) \in X^2$$

This can also be called max-min transitive. If this is not satisfied for some members of X, then $R(X, X)$ is nontransitive. If the given transitivity inequality is not satisfied for all the members $(x_i, x_k) \in X^2$, then the relation is called as antitransitive.

The max-product transitive can also be defined. It is given by

$$\mu_{\underset{\sim}{R}}(x_i, x_k) \geq \max_{x_j \in X} \cdot [\mu_{\underset{\sim}{R}}(x_i, x_j) \bullet \mu_{\underset{\sim}{R}}(x_j, x_k)] \quad \forall (x_i, x_k) \in X^2$$

The equivalence relation discussed can also be called similarity relation.

Fuzzy Tolerance Relation

A binary fuzzy relation that possesses the properties of reflexivity and symmetry is called fuzzy tolerance relation or resemblance relation. The equivalence relations are a special case of the tolerance relation. The fuzzy tolerance relation can be reformed into fuzzy equivalence relation in the same way as a crisp tolerance relation is reformed into crisp equivalence relation, i.e.,

$$\underbrace{\underset{\sim}{R}_1^{n-1}}_{\substack{\text{Fuzzy}\\ \text{tolerance}\\ \text{relation}}} = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 \circ \cdots \circ \underset{\sim}{R}_1 = \underbrace{\underset{\sim}{R}}_{\substack{\text{Fuzzy}\\ \text{equivalence}\\ \text{relation}}}$$

where "n" is the cardinality of the set that defines R_1.

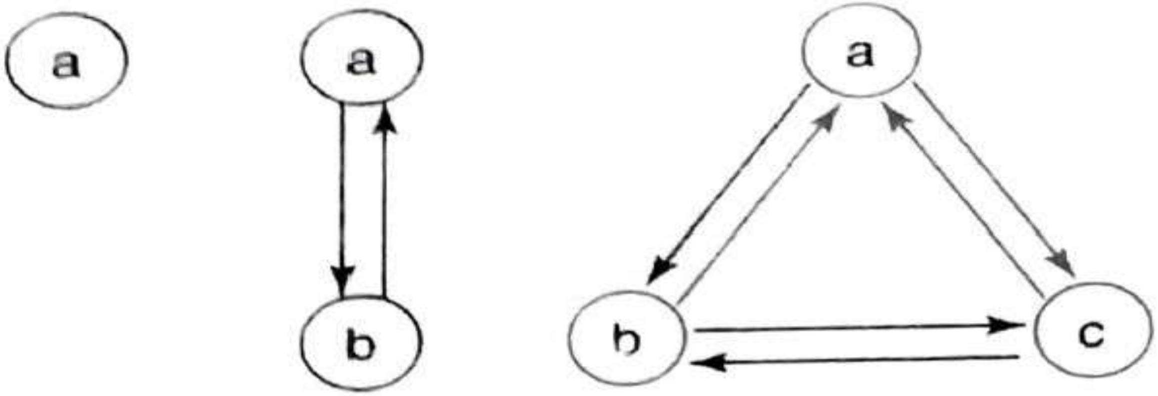
Which of the following are equivalence relations?

No.	Set	Relation on the set
(i)	People	is the brother of
(ii)	People	has the same parents as
(iii)	Points on a map	is connected by a road to
(iv)	Lines in plane geometry	is perpendicular to
(v)	Positive integers	for some integer k, equals 10^k times

Draw graphs of the equivalence relations.

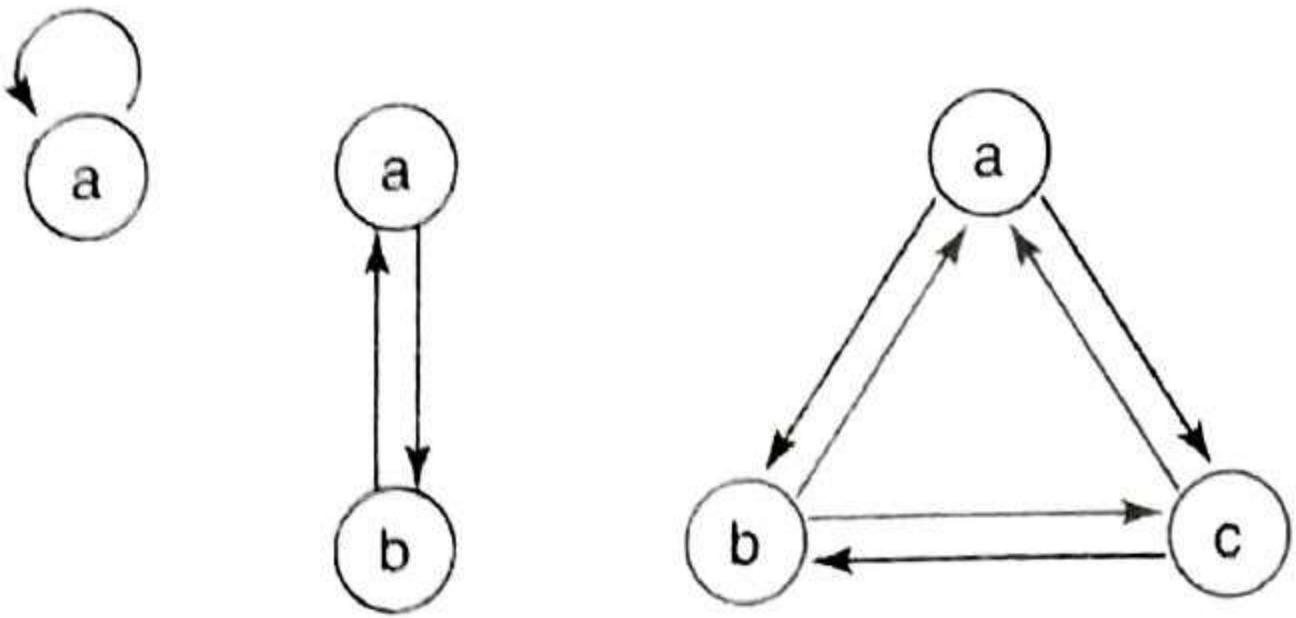
Solution:

(a) The set is people. The relation of the set "is the brother of." The relation (figure below) is not equivalence relation because people considered cannot be brothers to themselves. So, reflexive property is not satisfied. But symmetry and transitive properties are satisfied.



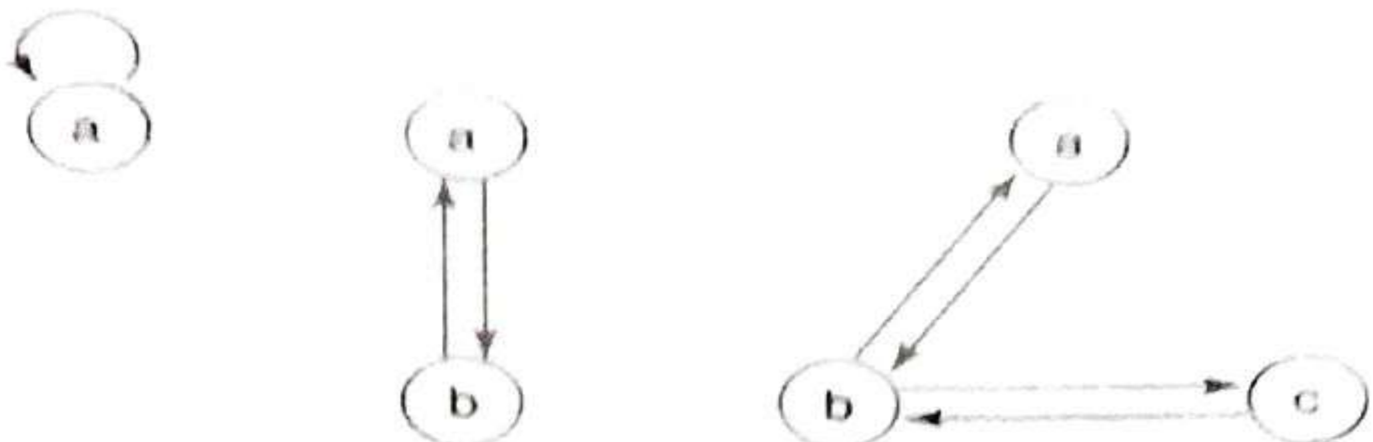
The figure illustrates that the relation is not an equivalence relation.

(b) The set is people. The relation is "has the same parents as." In this case (figure below), all the three properties are satisfied, hence it is an equivalence relation.



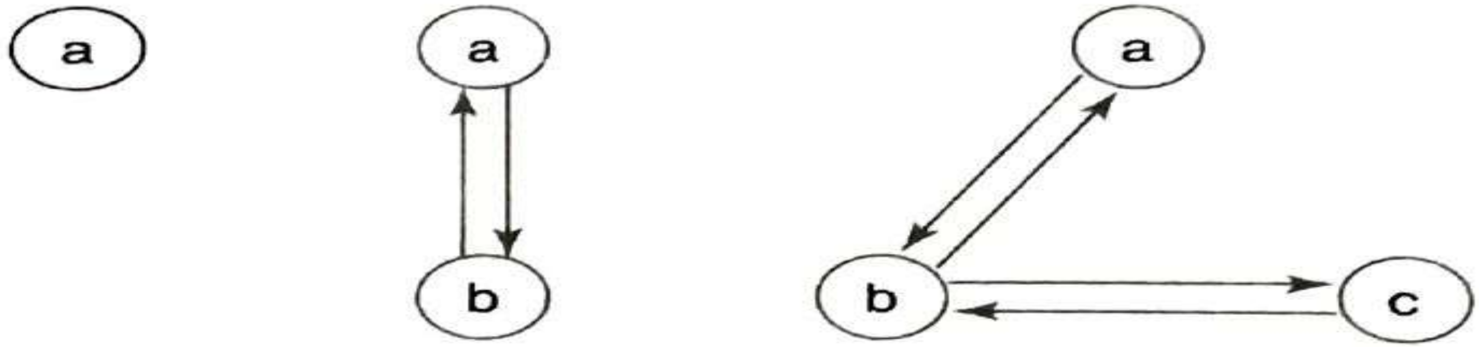
Thus the relation is an equivalence relation.

(c) The set is "points on a map." The relation is "is connected by a road to." This relation figure is not an equivalence relation because the transitive property is not satisfied. The road may connect 1st point and 2nd point; 2nd point and 3rd point; but it may not connect 1st and 3rd points. Thus, transitive property is not satisfied.



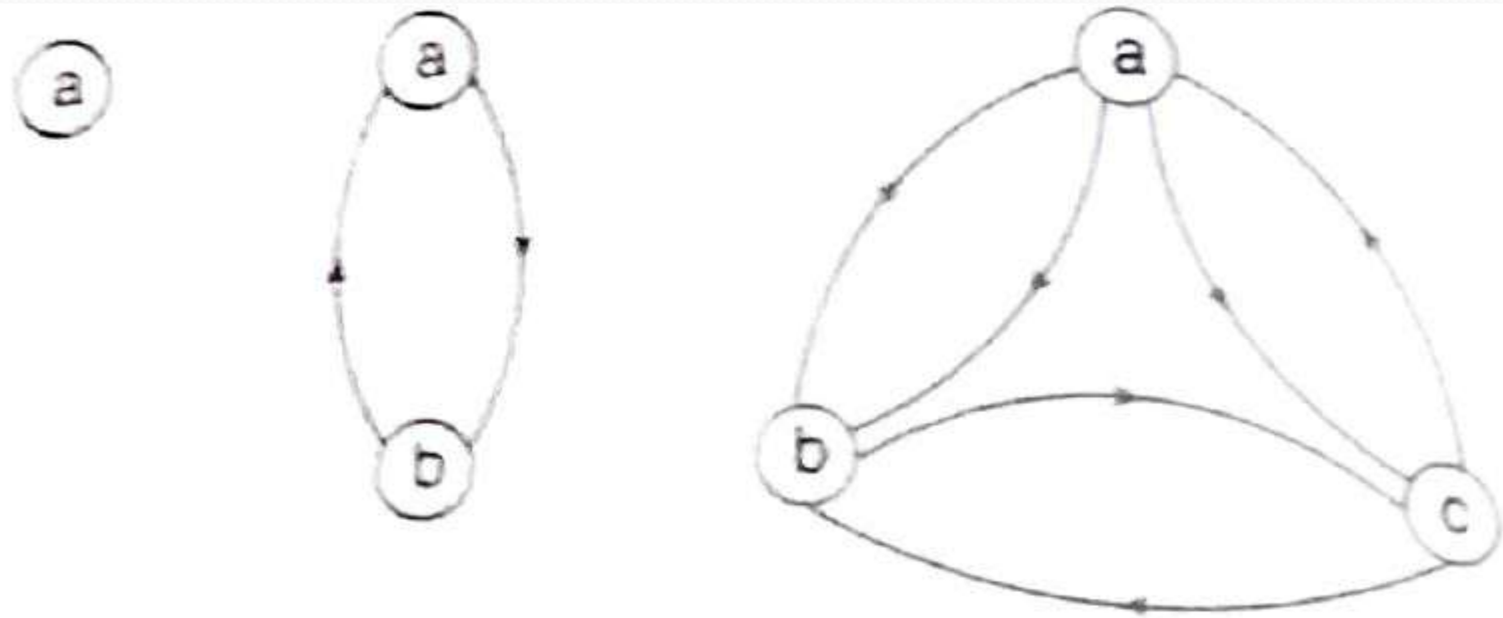
The figure illustrates that the relation is not an equivalence relation.

(d) The set is "lines in plane geometry." The relation "is perpendicular to." The relation (figure below) defined here is not an equivalence relation because both reflexive and transitive properties are not satisfied. A line cannot be perpendicular to itself, hence reflexivity is not satisfied. Also transitivity property is not satisfied because 1st line and 2nd line may be perpendicular to each other, 2nd line and 3rd line may also be perpendicular to each other, but 1st line and 3rd line will not be perpendicular to each other. However, symmetry property is satisfied.



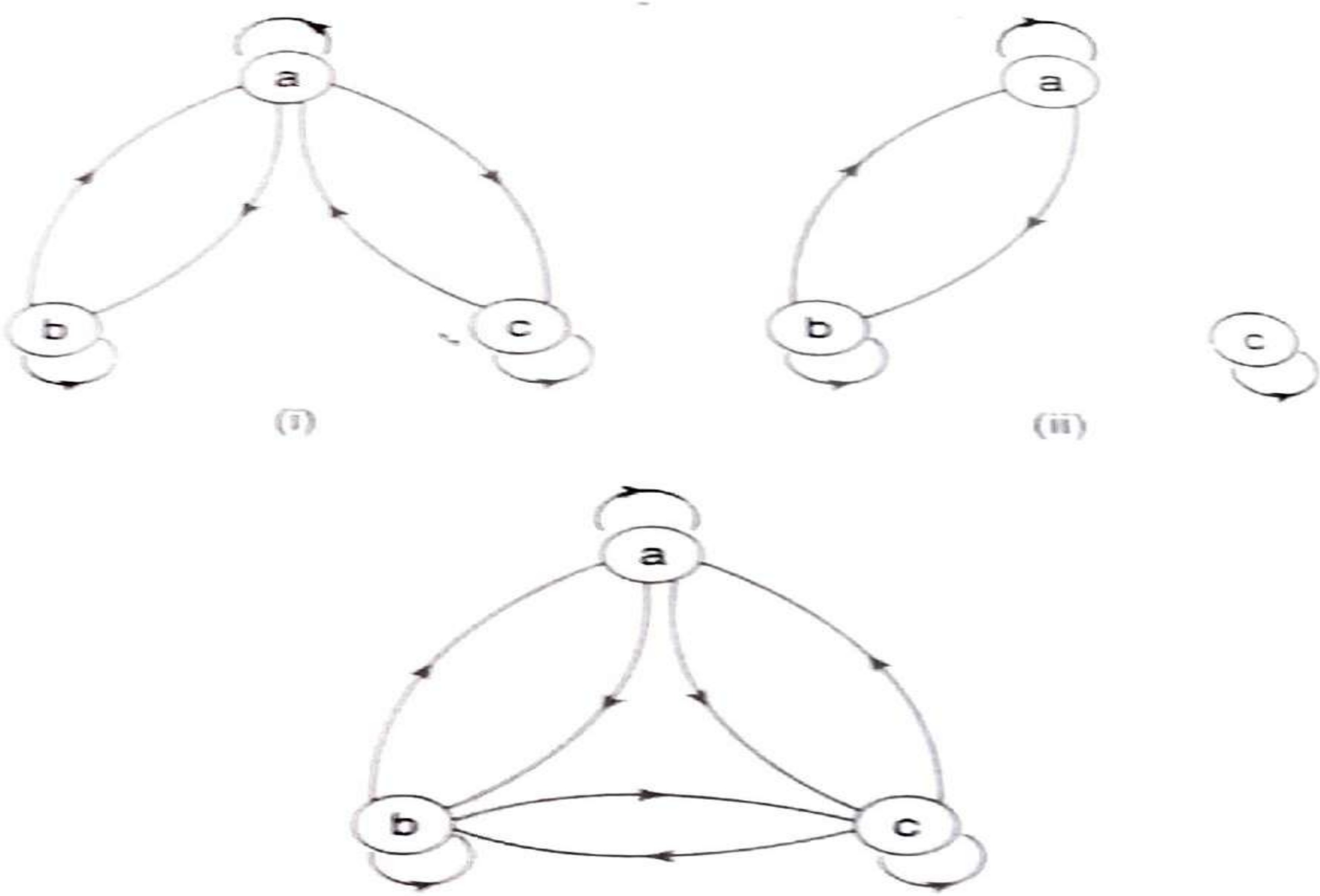
The figure illustrates that the relation is not an equivalence relation.

(e) The set is "positive integers". The relation is "for some integer k, equals 10^k times." In this case (figure below), reflexivity is not satisfied because a positive integer, for some integer k, equals 10^k times is not possible. Symmetry and transitivity properties are satisfied. Thus, the relation is not an equivalence relation.



The figure illustrates that the relation is not an equivalence relation.

10. The following figure shows three relations on the universe $X = \{a, b, c\}$. Are these relations equivalence relations?



(i) (ii) (iii)

Solution:

(a) The relation in (i) is not equivalence relation because transitive property is not satisfied.

(b) The relation in (ii) is not equivalence relation because transitive property is not satisfied.

(c) The relation in (iii) is equivalence relation because reflexive, symmetry and transitive properties are satisfied.

Equivalence Relations (Recap)

Crisp Equivalence Relations

A relation R on a universe X can also be thought of a relation from X to X. R i an equivalence relation if it has the following three properties (1) reflexivity (2 Symmetry and (3) Transitivity

1. Reflexivity: $(x_i, x_i) \in R$ or $\chi_R(x_i, x_i) = 1$
2. Symmetry: $(x_i, x_j) \in R \rightarrow (x_j, x_i) \in R$

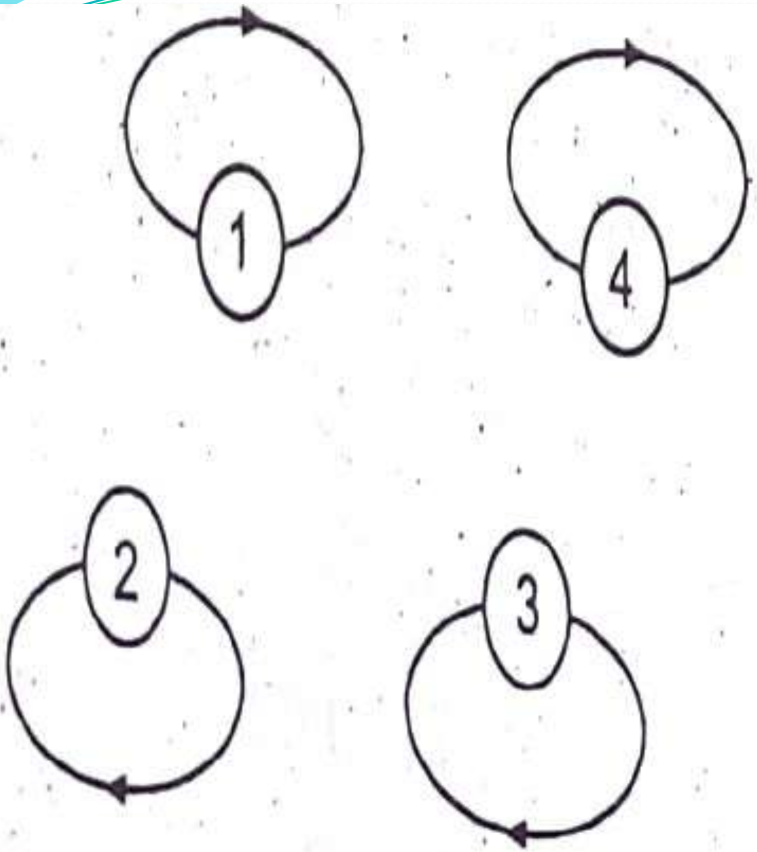
   (or)

   $\chi_R(x_i, x_j) = \chi_R(x_j, x_i)$
3. Transitivity : $(x_i, x_j) \in R$ and $(x_j, x_k) \in R \rightarrow (x_i, x_k) \in R$

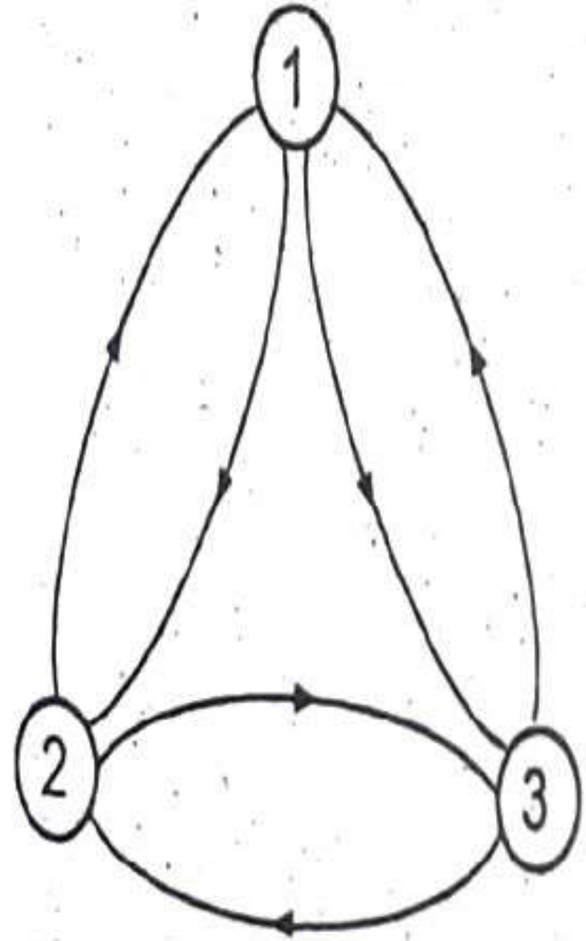
   (or)

   $\chi_R(x_i, x_j)$ and $\chi_R(x_j, x_k) = 1$

   $\rightarrow \chi_R(x_i, x_k) = 1$



(a) *Reflexivity*



(b) *Symmetry and transitivity*

Crisp Tolerance Relation

Crisp tolerance relation

A tolerance relation R (also called a proximity relation) on a universe X is a relation that exhibits only the properties of reflexivity and symmetry. A tolerance relation R can be reformed into an equivalence relation by (n – 1) compositions with itself where n is the cardinal number of the set.

Consider the following

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The above relation is reflexive and symmetric but not transitive

$$(x_2, x_3) \in R_1, (x_3, x_4) \in R_1 \text{ but } (x_2, x_4) \notin R_1$$

For it to become an equivalence relation

$$R_1 \circ R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} x_1 & R & x_2 & x_3 & \\ x_2 & R & x_1 & x_3 & x_4 \\ x_3 & R & x_1 & x_2 & x_4 \\ x_4 & R & x_2 & x_3 & \\ x_5 & R & x_5 & & \end{matrix}$$

hence the above relation is symmetric, transitive and reflexive.

Fuzzy Tolerance & Equivalence Relations

A fuzzy relation R on a single universe X is a relation from X to X. It is a fuzzy equivalence relation if all three of the following properties for matrix relations define it.

Reflexivity: $\mu_R(x_i, x_i) = 1$

Symmetry: $\mu_R(x_i, x_j) = \mu_R(x_j, x_i)$

Transitivity: $\mu_R(x_i, x_j) = \lambda_1$ and $\mu_R(x_j, x_k) = \lambda_2$

$\lambda_R(x_i, x_k) = \lambda$ where $\lambda \geq \min[\lambda_1, \lambda_2]$

Problem

Any fuzzy relation that has the properties of reflexivity and symmetry is tolerance relation and can be transformed into a fuzzy equivalence relation by the most (n – 1) compositions.

Consider the fuzzy relation

$$R = \begin{bmatrix} 1 & 0.4 & 0.1 & 0 & 0.5 \\ 0.4 & 1 & 0.5 & 0 & 0.3 \\ 0.1 & 0.5 & 1 & 0.6 & 0.7 \\ 0 & 0 & 0.6 & 1 & 0.4 \\ 0.5 & 0.3 & 0.7 & 0.4 & 1 \end{bmatrix}$$

$$\mu_R(x_1, x_2) = 0.4 \; \mu_R[x_2, x_3] = 0.5 > 0.4$$

$$\mu_R(x_1, x_3) = 0.1 \quad < \min(0.4, 0.5)$$

$$< 0.4$$

hence the above relation is not transitive

$$R \circ R = \begin{bmatrix} 1 & 0.4 & 0.1 & 0 & 0.5 \\ 0.4 & 1 & 0.5 & 0.5 & 0.4 \\ 0.1 & 0.5 & 1 & 0.6 & 0.7 \\ 0 & 0 & 0.6 & 1 & 0.4 \\ 0.5 & 0.3 & 0.7 & 0.4 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0.4 & 0.1 & 0 & 0.5 \\ 0.4 & 1 & 0.5 & 0 & 0.3 \\ 0.1 & 0.5 & 1 & 0.6 & 0.7 \\ 0 & 0 & 0.6 & 1 & 0.4 \\ 0.5 & 0.3 & 0.7 & 0.4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.4 & 0.5 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 & 0.5 & 0.4 \\ 0.5 & 0.5 & 1 & 0.6 & 0.7 \\ 0.4 & 0.5 & 0.6 & 1 & 0.6 \\ 0.5 & 0.4 & 0.7 & 0.6 & 1 \end{bmatrix}$$

Hence the above relation is reflexive symmetric and transitive.