SOFT COMPUTING MODULE 4

FUZZY Membership Functions

Topics

- Fuzzy membership functions
- Features of membership function
- Fuzzification
- Methods of membership value assignments
- λ- Cut or α- Cut
- Defuzzification

Membership functions

- *Membership function* defines the fuzziness in a fuzzy set irrespective of the elements in the set, which are discrete or continuous.
- They are generally represented in graphical form.
- The rules that describe fuzziness graphically are also fuzzy.

- The membership function defines all the information contained in a fuzzy set.
- A fuzzy set A in the universe of discourse X can be defined a set of ordered pairs:

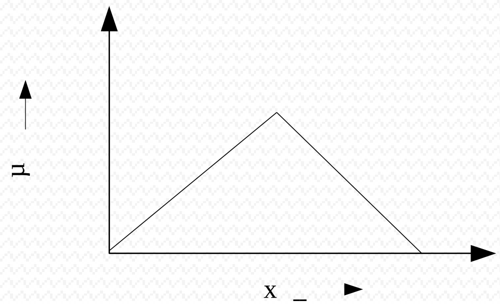
$$A = \{(x, \mu_A(x)) | x \in X\}$$

where, $\mu_A(.)$ is called membership function of A.

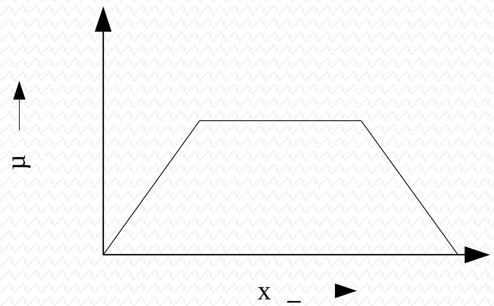
Fuzzy membership functions

Membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

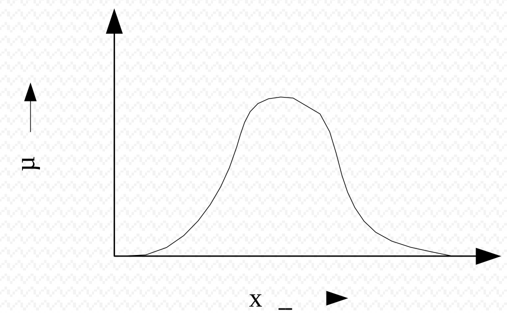
Following figures shows a typical examples of membership functions.



< triangular >



< trapezoidal >



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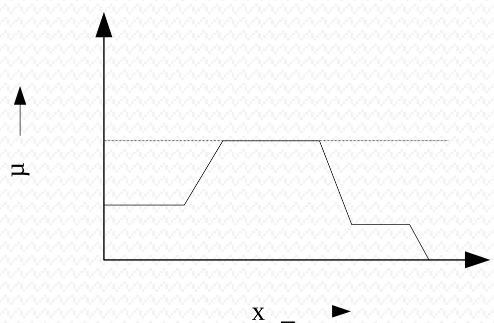
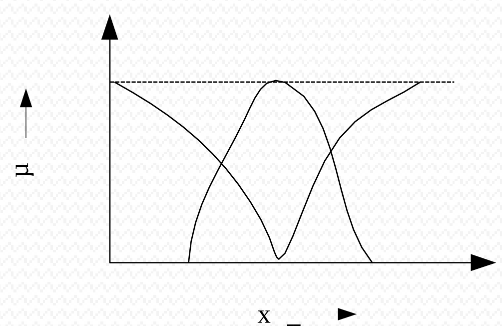


< non-uniform >



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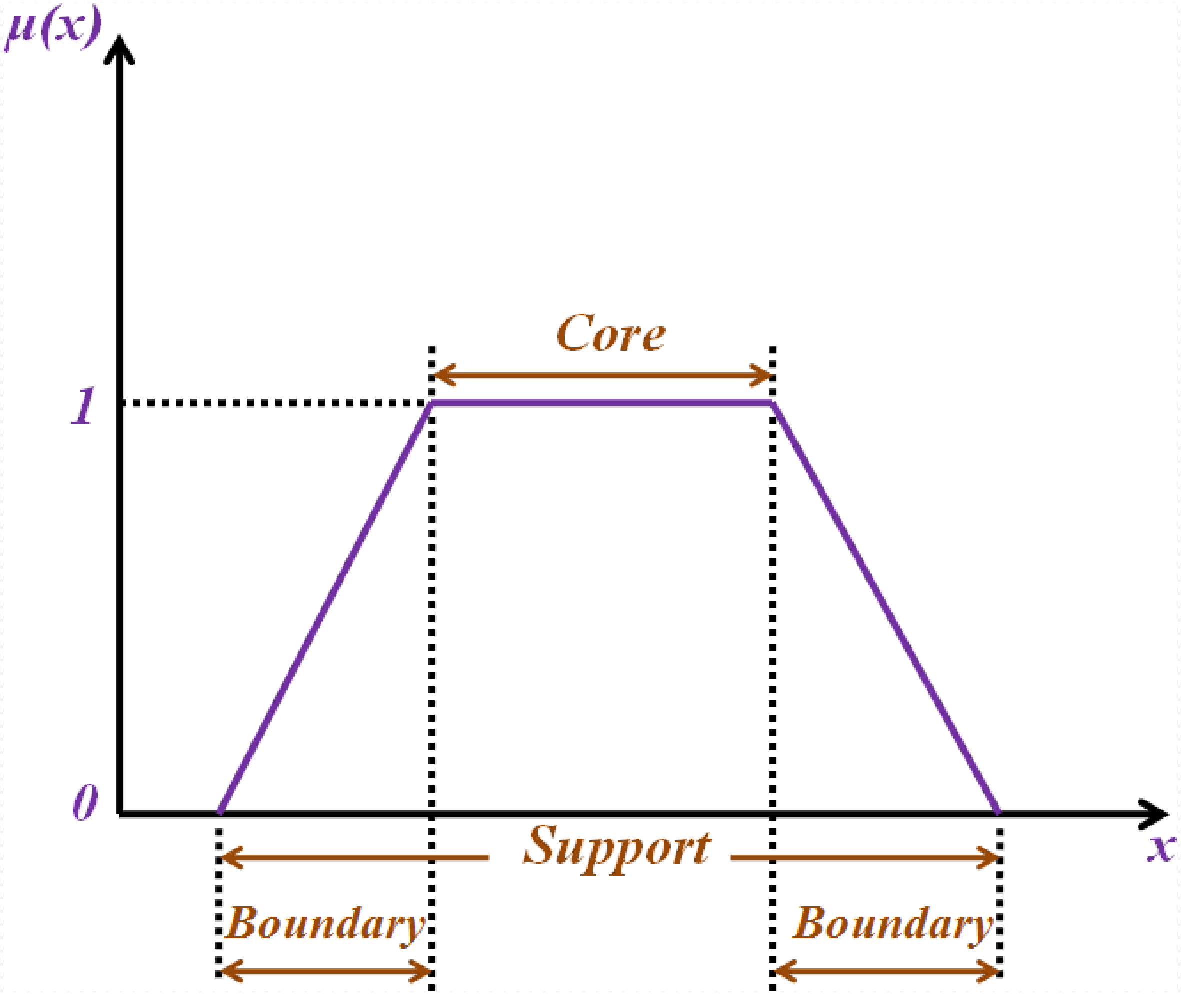
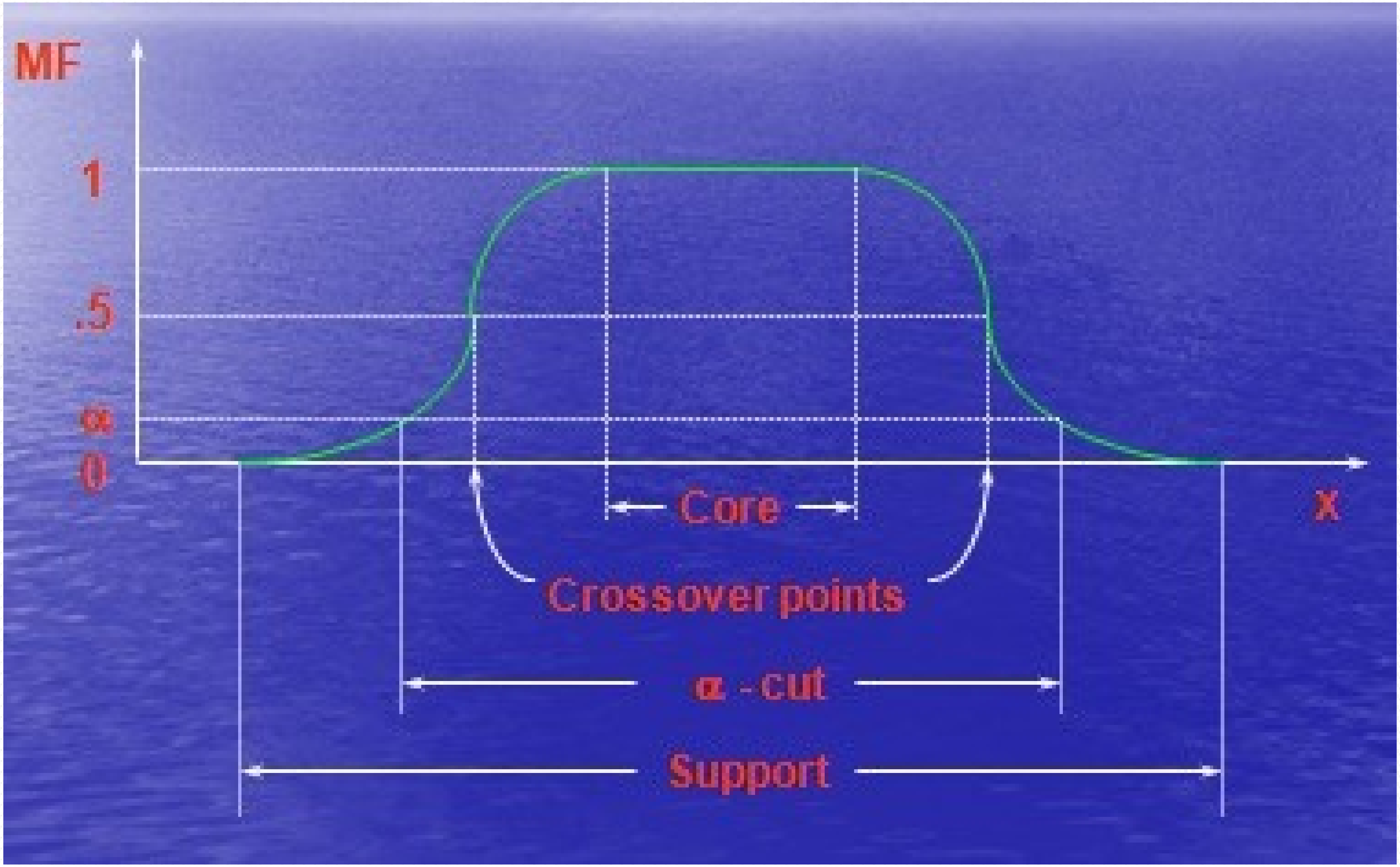
Features of membership funcion




MF
1
.5
α
0
X
Core
Crossover points
α - cut
Support

Core

- The *core* of a membership function for some fuzzy set A is defined as that region of universe is characterized by complete membership in the set $A.ie$,

$$\mu_A(x) = 1$$

- The core of a fuzzy set may be an empty set.

Support

- The *support* of a membership function for a fuzzy set A is defined as that region of universe is characterized by a nonzero membership in the set $A. ie$,

$$\mu_A(x) > 0$$

- A fuzzy set whose support is a single element in X with

$$\mu_A(x) = 1$$

is referred to as a fuzzy singleton.

Boundary

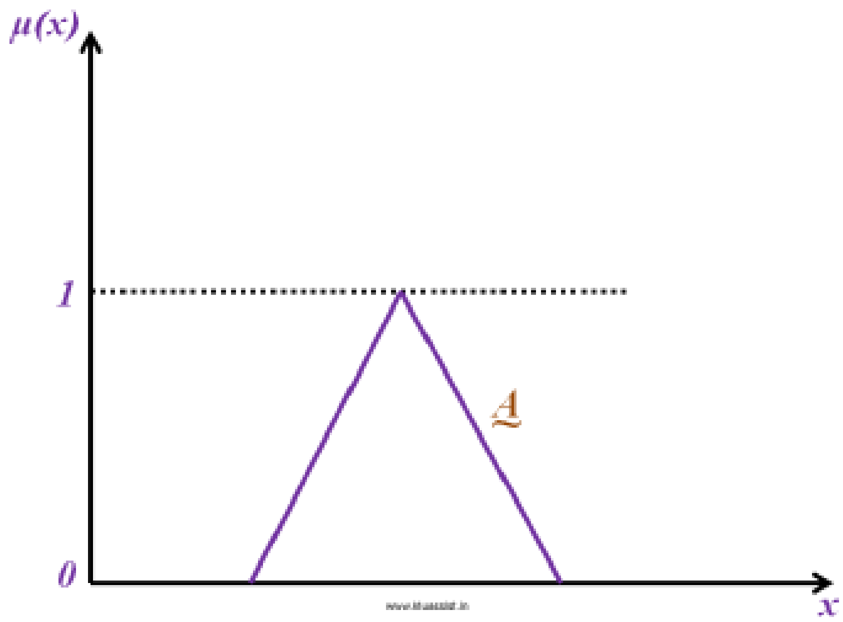
- The *boundary* of a membership function for a fuzzy set A is defined as that region of universe containing elements that have a nonzero but not complete membership.*ie*,

$$0 < \mu_A(x) < 1$$

- The boundary elements are those which possess partial membership in the fuzzy set A.

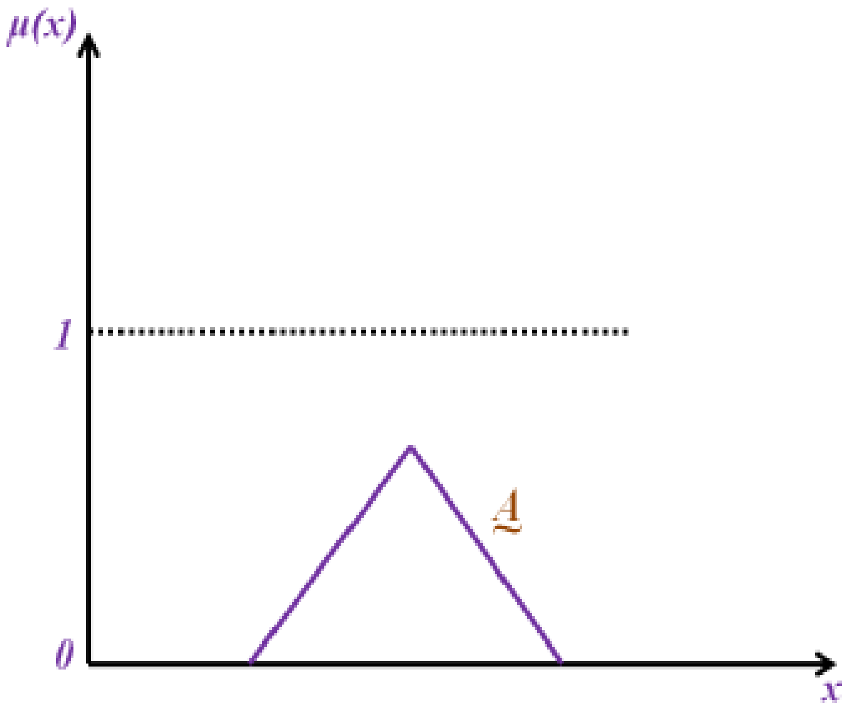
Normal fuzzy set

- A fuzzy set whose membership function has at least one element x in the universe whose membership value is unity is called *normal fuzzy set*.
- The element for which the membership is equal to *1* is called *prototypical element.*



Subnormal fuzzy set

- A fuzzy set where in no membership function has its value equal to *1* is called *subnormal fuzzy set*.

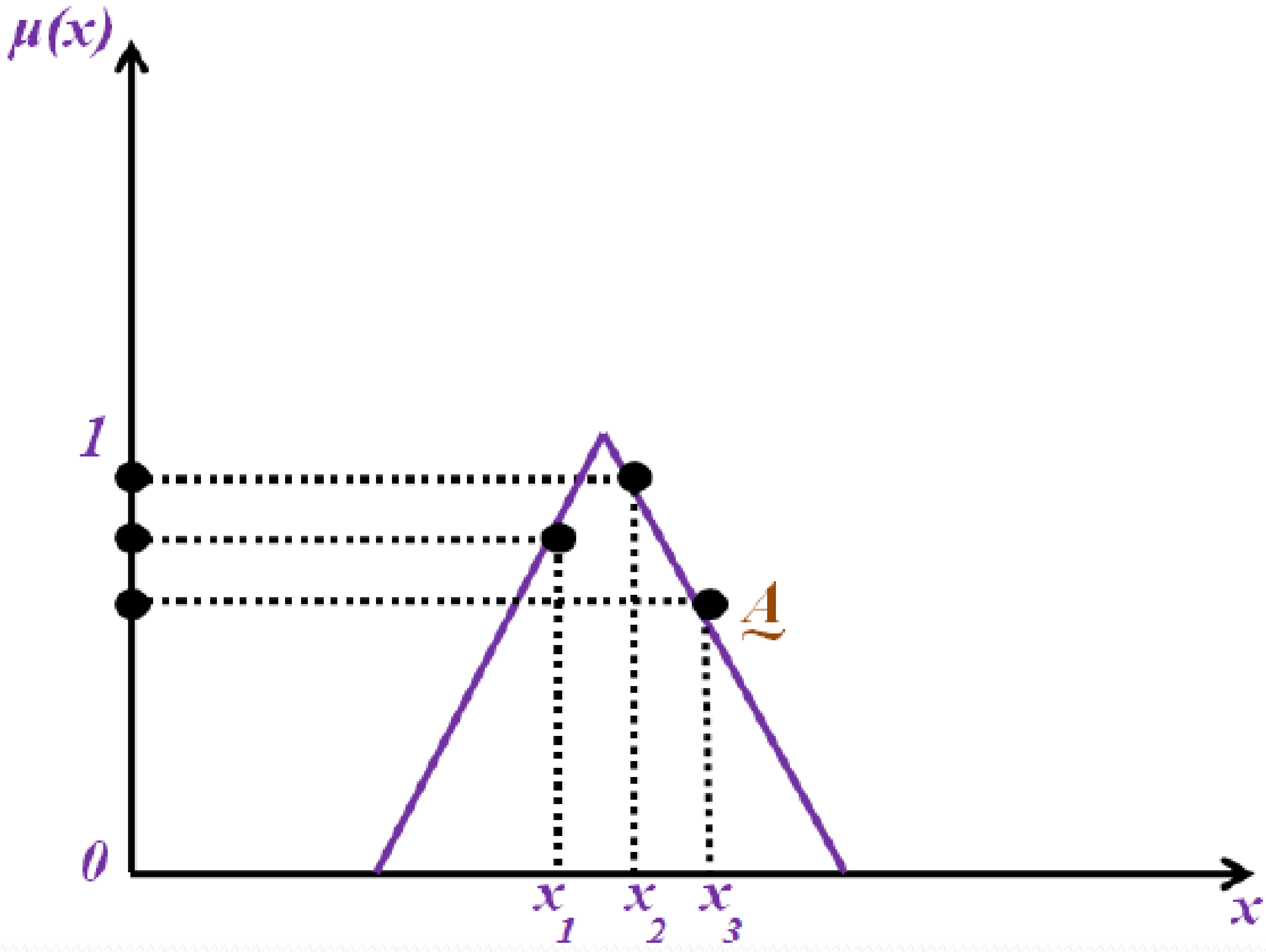


Convex fuzzy set

- A *convex fuzzy set* has a membership function whose membership values are strictly monotonically increasing or strictly monotonically decreasing or strictly monotonically increasing than strictly monotonically decreasing with increasing values for elements in the universe.
- For elements x_1, x_2 and x_3 in a fuzzy set A. If,

$$\mu_A(x_2) \geq min[\mu_A(x_1), \mu_A(x_3)]$$

  then A is said to be convex fuzzy set.
- The intersection between two convex fuzzy sets is also a convex fuzzy set.
- A fuzzy set possessing characteristics opposite to that of convex fuzzy set is called *non convex fuzzy set*.



μ(x)
1
0
A
x
x
1
x
2
x
3

Non convex

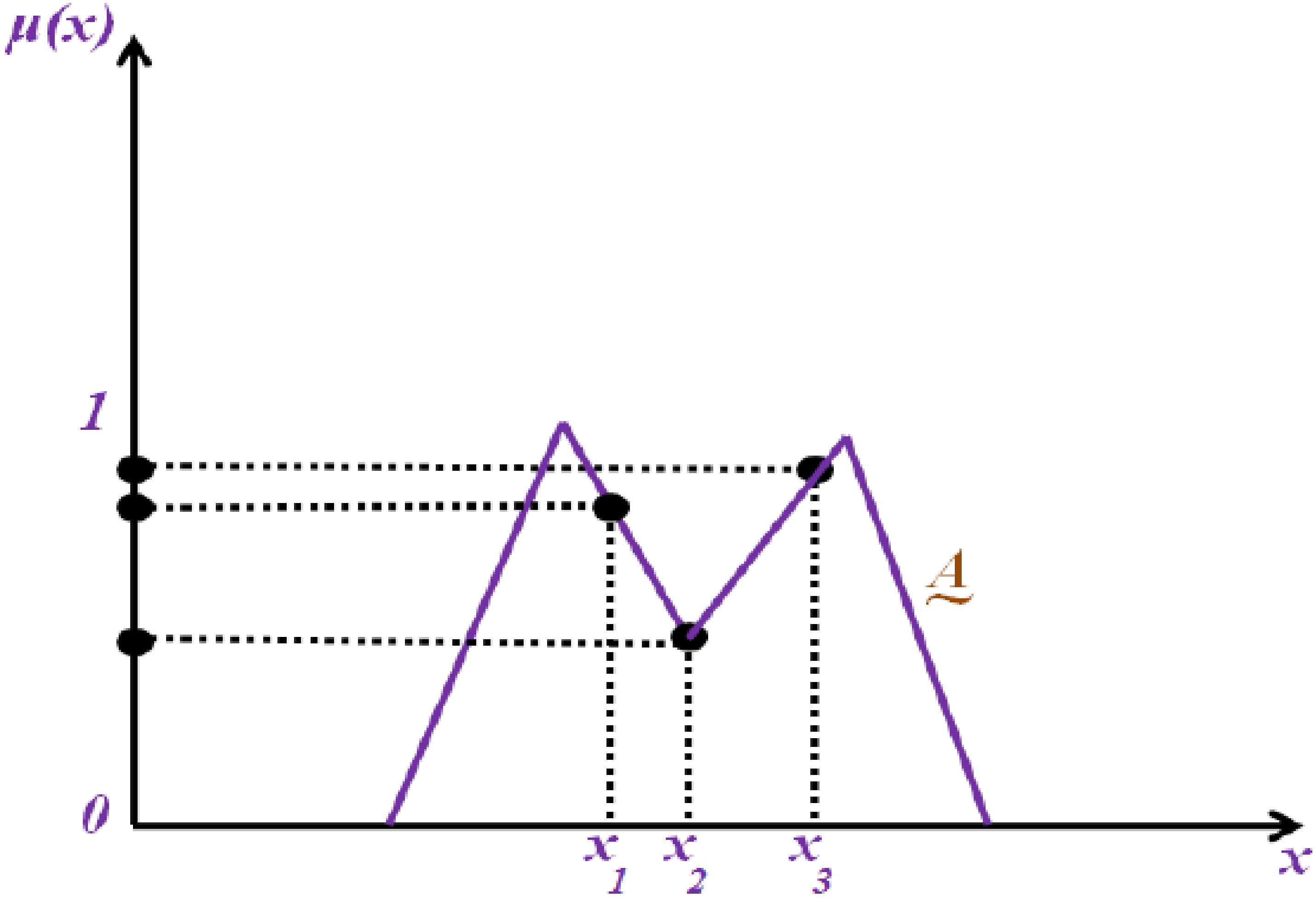


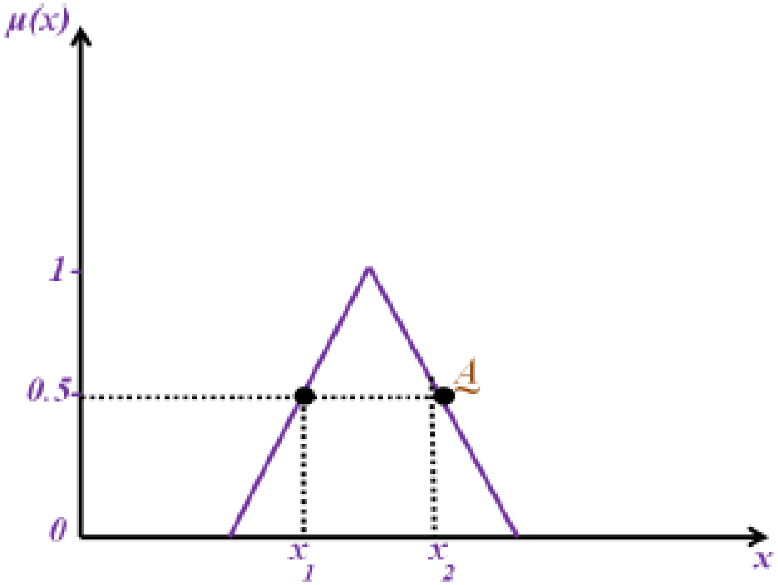
Figure 2.2: Non convex normal fuzzy set

Crossover point of a fuzzy set

- The element in the universe for which a particular fuzzy set A has its value equal to *0.5* is called *crossover point* of a membership function.*ie*,

$$\mu_A(x) = 0.5$$

- There can be more than one crossover point in a fuzzy set.



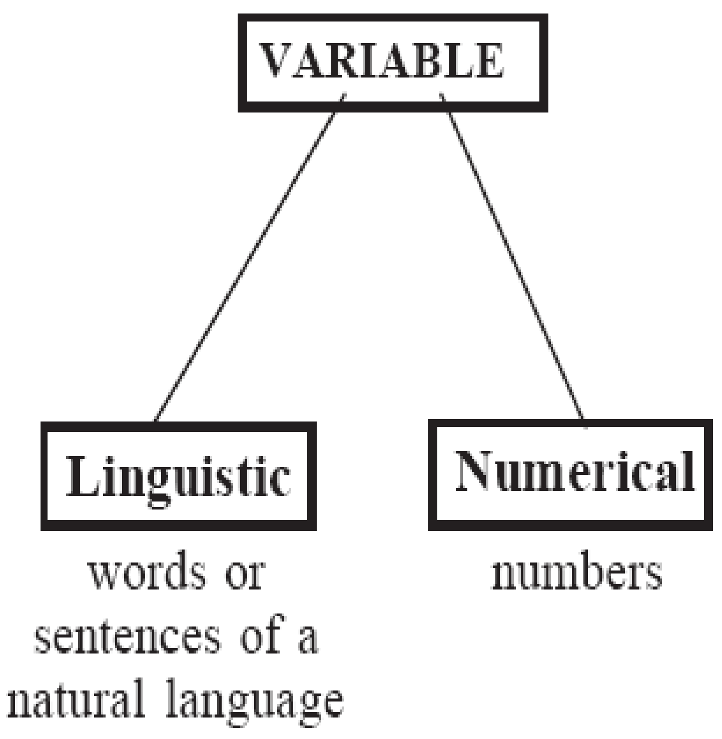
Height of the fuzzy set

- The maximum value of the membership function in a fuzzy set A is called as the *height* of the fuzzy set.
- For a normal fuzzy set, the height is equal to *1*.
- If the height of a fuzzy set is less than *1*, then the fuzzy set is called subnormal fuzzy set.
- When the fuzzy set A is a convex single point normal fuzzy set, then A is termed as a fuzzy number.

Linguistic variables and hedges

Linguistic Variable

- A linguistic variable is one with a value that is a natural language expression referring to some quantity of interest.
- *So the value of the linguistic variable is a word* or a sentence. This is the main difference between a linguistic variable and a numerical one.



Linguistic variables and hedges

Linguistic Variable

- Generally, a linguistic variable is a composite term u = u1, u2, ...,un which is a concatenation of ***atomic terms*** u1, u2, ...,un
- atomic terms can be divided into four categories:
  - **primary terms**, which are the labels of specified fuzzy subsets of the universe of discourse (e.g. small and big);
  - **Connectives:** AND, OR and the negation NOT;
  - **hedges:** such as VERY, MOST, RATHER, SLIGHTLY, MORE OR LESS, etc.;
  - **markers** such as parenthesis.

Linguistic variables and hedges

primary terms

- A primary term is actually just a name or a label of a fuzzy set. It usually describes the word which is used by experts to express their opinion about the value of one of the object characteristics, e.g : old, large, fast, etc.

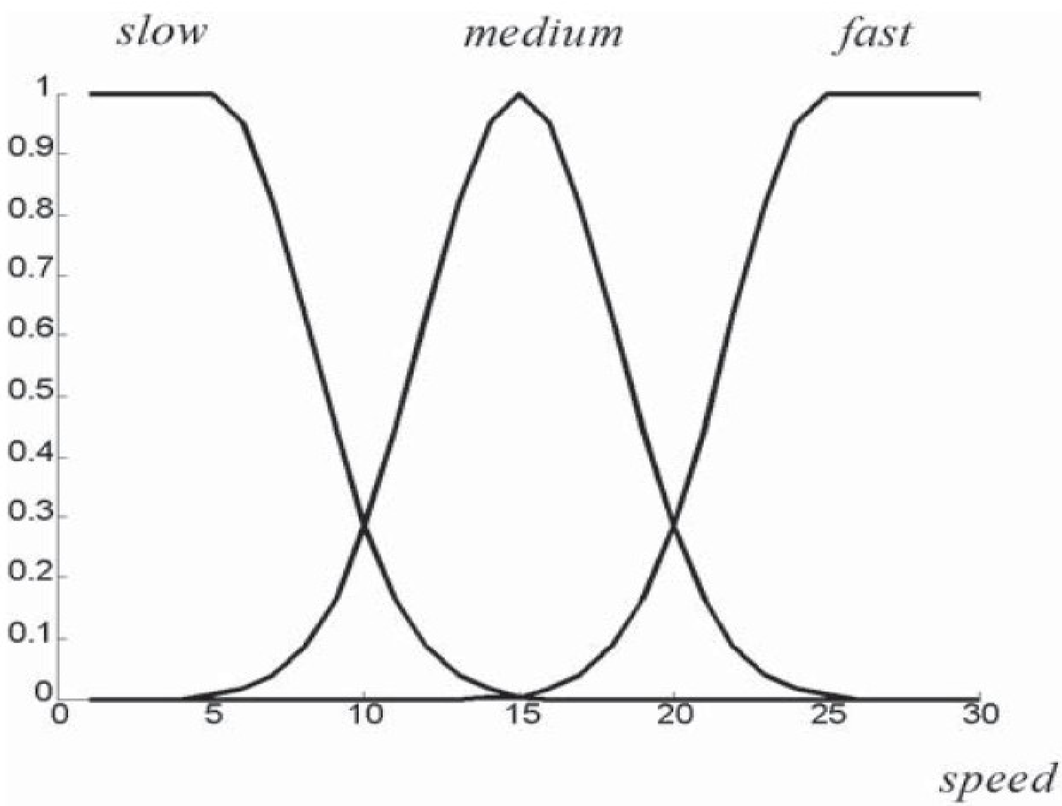
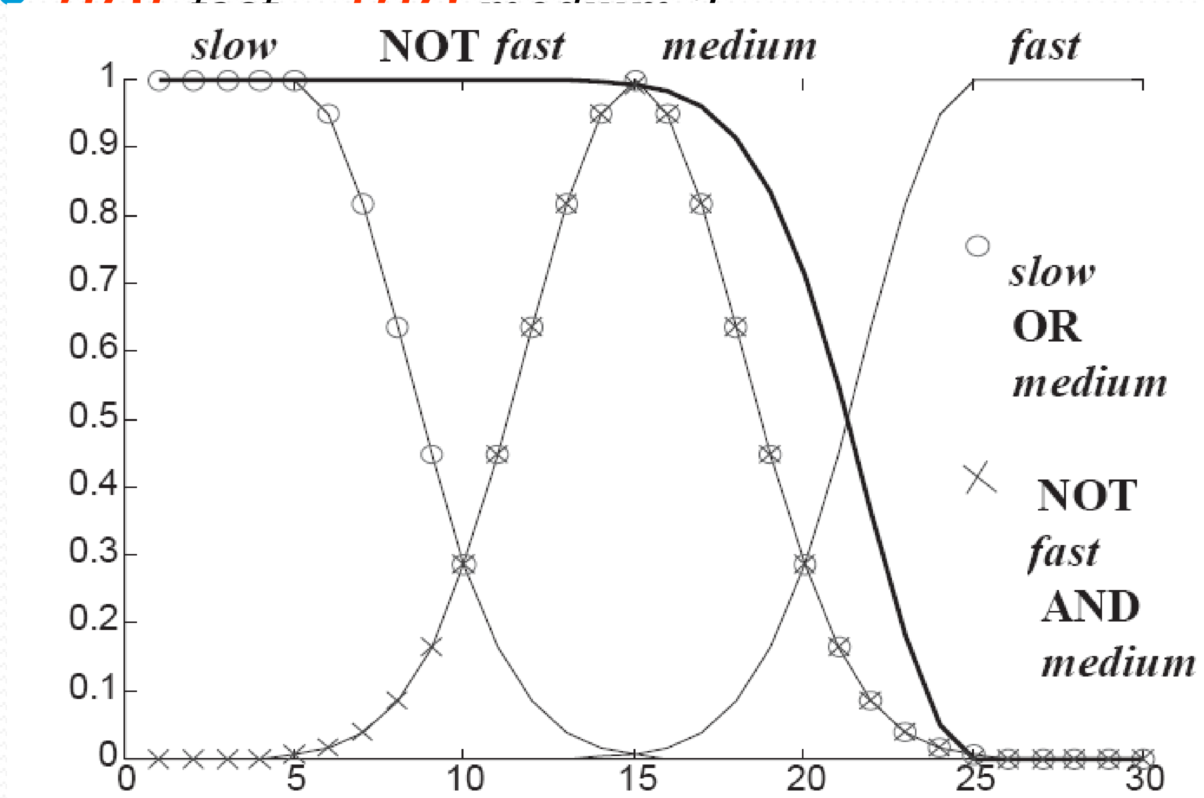


Fig. 2.22 *Membership functions for linguistic values*

Linguistic variables and hedges

connectives

- The connectives (AND, OR, and NOT) realise the operations of intersections, union, complement considered earlier.
- ***Example :****How can we express*
  - *slow* ***or*** *medium ?*
  - *not fast and medium ?*



Linguistic variables and hedges

hedges(modifiers)

- Hedges are used to produce a larger set of values for a linguistic variable from a small collection of primary terms through the processes of intensification or concentration, dilation and fuzzification.
- For example, the operator '*very*' is usually defined as a concentration operator as:

This is one way not always

$$\textit{very}\ u = u^2$$

- This operator can also be composed with itself, thus:

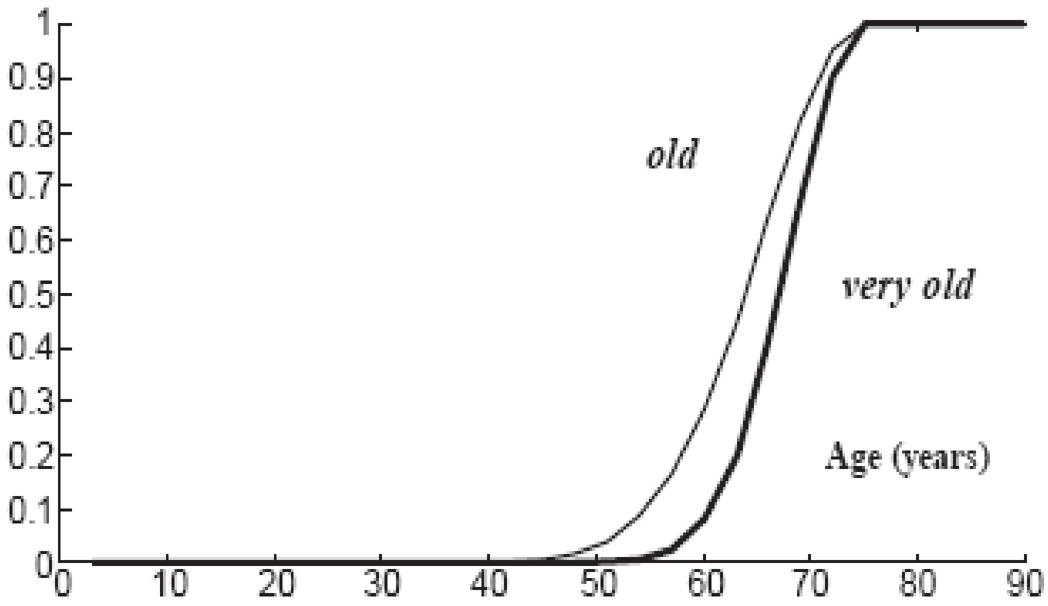
$$\textit{very}\ (\text{very } u) = (\text{very } u)^2 = u^4$$

Linguistic variables and hedges

hedges (modifiers)

- Hedges (example): *The composite term 'very old' can be obtained from the term old as*

  $$\textit{very}\ \text{old} = \text{old}^2$$



Example : consider a fuzzy set

$$young = \left\{ \frac{0.8}{20} \quad \frac{0.6}{30} \quad \frac{0.2}{40} \quad \frac{0}{50} \right\}$$

$$\Downarrow$$

$$very\ \ young = \left\{ \frac{0.64}{20} \quad \frac{0.36}{30} \quad \frac{0.04}{40} \quad \frac{0}{50} \right\}$$

$$\Downarrow$$

$$very\ very\ \ young = \left\{ \frac{0.4096}{20} \quad \frac{0.1296}{30} \quad \frac{0.0016}{40} \quad \frac{0}{50} \right\}$$

Another example

Extremely A = A^3

Linguistic variables and hedges

- Wind is *a little* strong.
- Weather is *quite* cold.
- Height is *almost* tall.
- Weight is *very* high.
- Wind, Weather, Height and Weight are linguistic variables.
- A little, Quite, Almost, Very are hedges.
- Strong, Cold, Tall and high are linguistic value.

Example

if $\underbrace{\text{temperature}}_{\text{Linguistic Variable}}$ is $\underbrace{\text{cold}}_{\text{Linguistic Value}}$ and $\underbrace{\text{oil}}_{\text{Linguistic Variable}}$ is $\underbrace{\text{cheap}}_{\text{Linguistic Value}}$

then $\underbrace{\text{heating}}_{\text{Linguistic Variable}}$ is $\underbrace{\text{high}}_{\text{Linguistic Value}}$

Definition [Zadeh 1973]

A linguistic variable is characterized by a quintuple

$$\left(x, T(x), U, G, M \right)$$

Name

Term Set

Universe

Syntactic Rule

Semantic Rule

Example

A linguistic variable is characterized by a quintuple

$$\left(x, T(x), U, G, M \right)$$

age

$$G(\text{age}) = \left\{ \begin{array}{l} \text{old, very old, not so old,} \\ \text{more or less young,} \\ \text{quite young, very young} \end{array} \right\}$$

[0, 100]

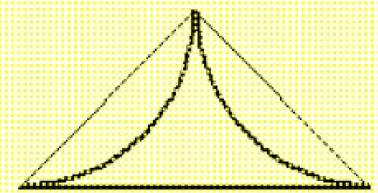
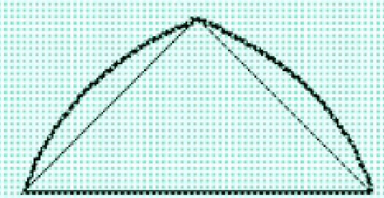
Example semantic rule:

$$M(\text{old}) = \left\{ \left(u, \mu_{\text{old}}(u) \right) \middle| u \in [0, 100] \right\}$$

$$\mu_{\text{old}}(u) = \begin{cases} 0 & u \in [0, 50] \\ \left[1 + \left(\dfrac{u - 50}{5} \right)^{-2} \right]^{-1} & u \in [50, 100] \end{cases}$$

Example

- Membership of body fitness

Hedge	*Mathematical Expression*	*Graphical Representation*
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	

Fuzzification

- *Fuzzification* is the process of transforming a crisp set to a fuzzy set.
- This operation translates accurate crisp input values into linguistic variables.
- They possess uncertainity within themselves.
- The variable is probably fuzzy and can be represented by a membership function.

Kernel of fuzzification

- For a fuzzy set,

$$A = \{\frac{\mu_i}{x_i} | x_i \in X\}$$

- A common fuzzification algorithm is performed by keeping μ_i constant and x_i being transformed to a fuzzy set $Q(x_i)$ depicting the expression about x_i.
- The fuzzy set $Q(x_i)$ is referred to as the *Kernel of fuzzification or support fuzzification or s–fuzzification*.
- The fuzzified set A can be expressed as,

$$\mu_1 Q(x_1) + \mu_2 Q(x_2) + ... + \mu_n Q(x_n)$$

- *Grade fuzzification or g–fuzzification*: where x_i is kept constant and μ_i is expressed as a fuzzy set.

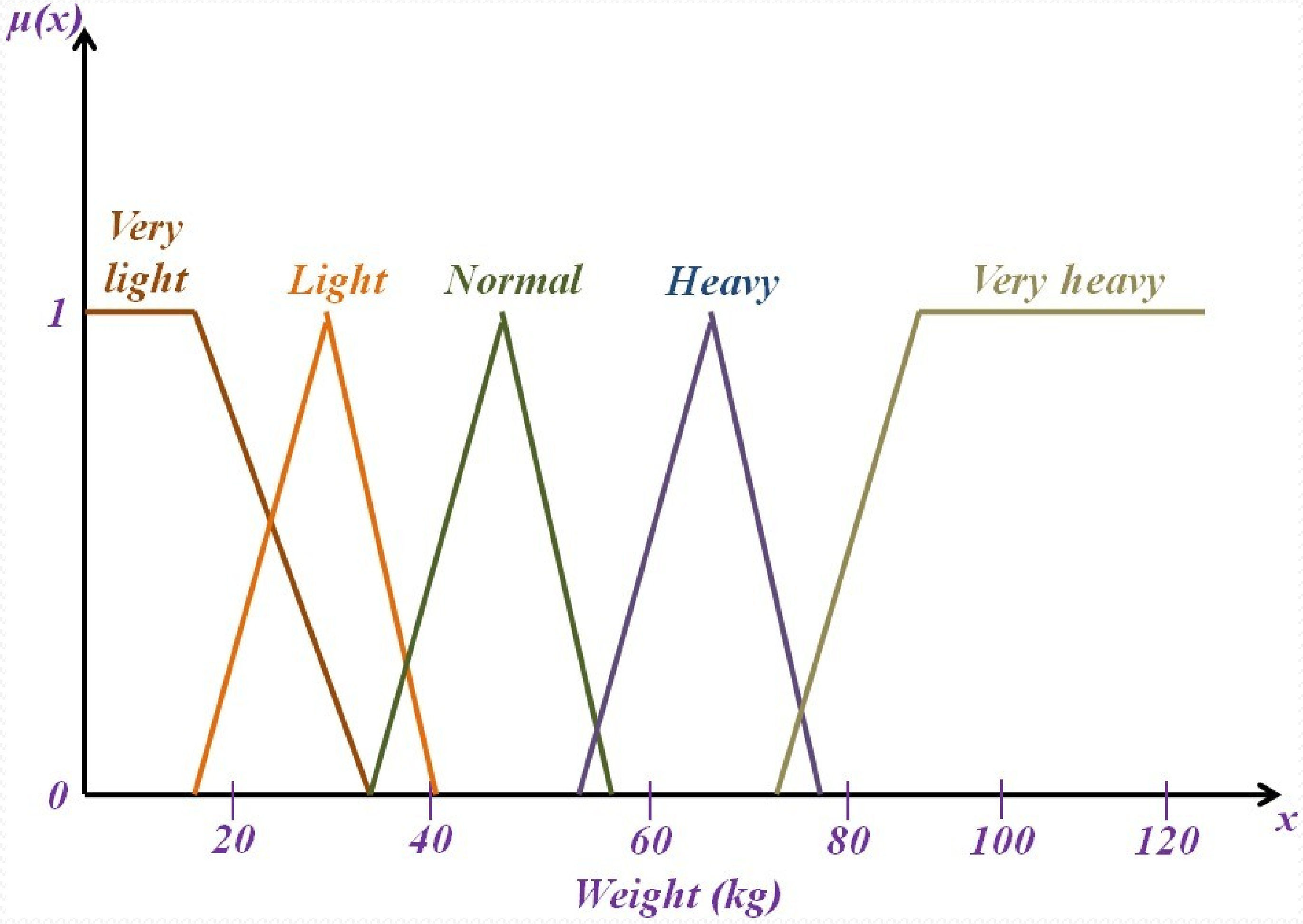
Methods of membership value assignments

1. Intuition
2. Inference
3. Rank ordering
4. Angular fuzzy sets
5. Neural networks
6. Genetic algorithm
7. Inductive reasoning

Intuition

- *Intuition method* is based upon the common intelligence of humans.
- It is the capacity of the human to develop membership func- tions on the basis of their own intelligence and understanding capability.
- There should an in depth knowledge of the application to which membership value assignment has to be made.

Membership functions for the fuzzy variable "weight"



Problems

(1) *Using your own intuition and definitions of the uni- verse of discourse, plot fuzzy membership functions for* "*weight of people*".

$U = weight\ of\ people$

$Let\ the\ weights\ be\ in\ kilogram.$

$Let\ the\ linguistic\ variables\ be\ the\ following:$

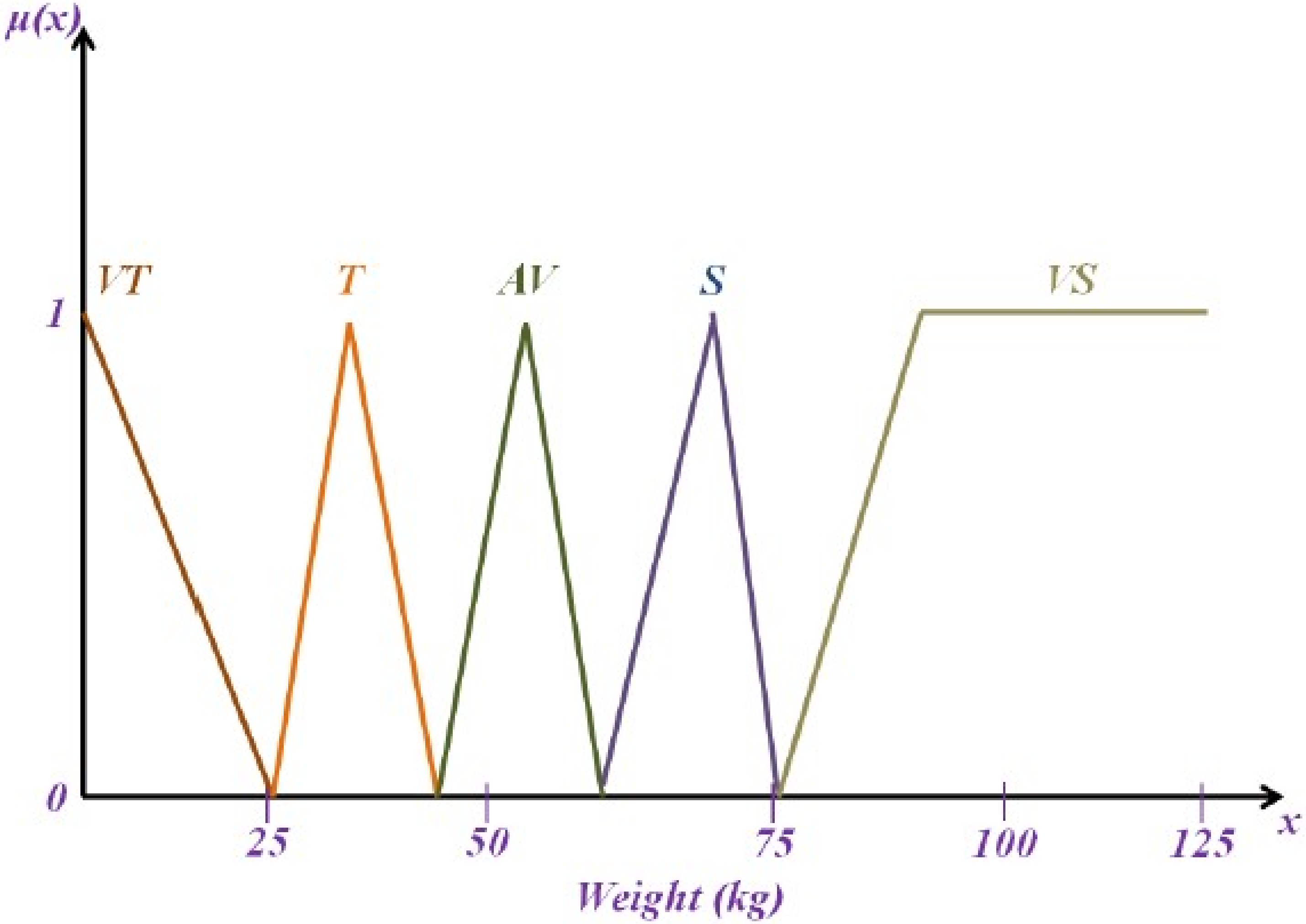
$$Very\ thin(VT) : W \leq 25$$

$$Thin(T) : 25 < W \leq 45$$

$$Average(AV) : 45 < W \leq 60$$

$$Stout(S) : 60 < W \leq 75$$

$$Very\ stout(VS) : W > 75$$


μ(x)
1
0
VT
T
AV
S
VS
25
50
75
100
125
x
Weight (kg)

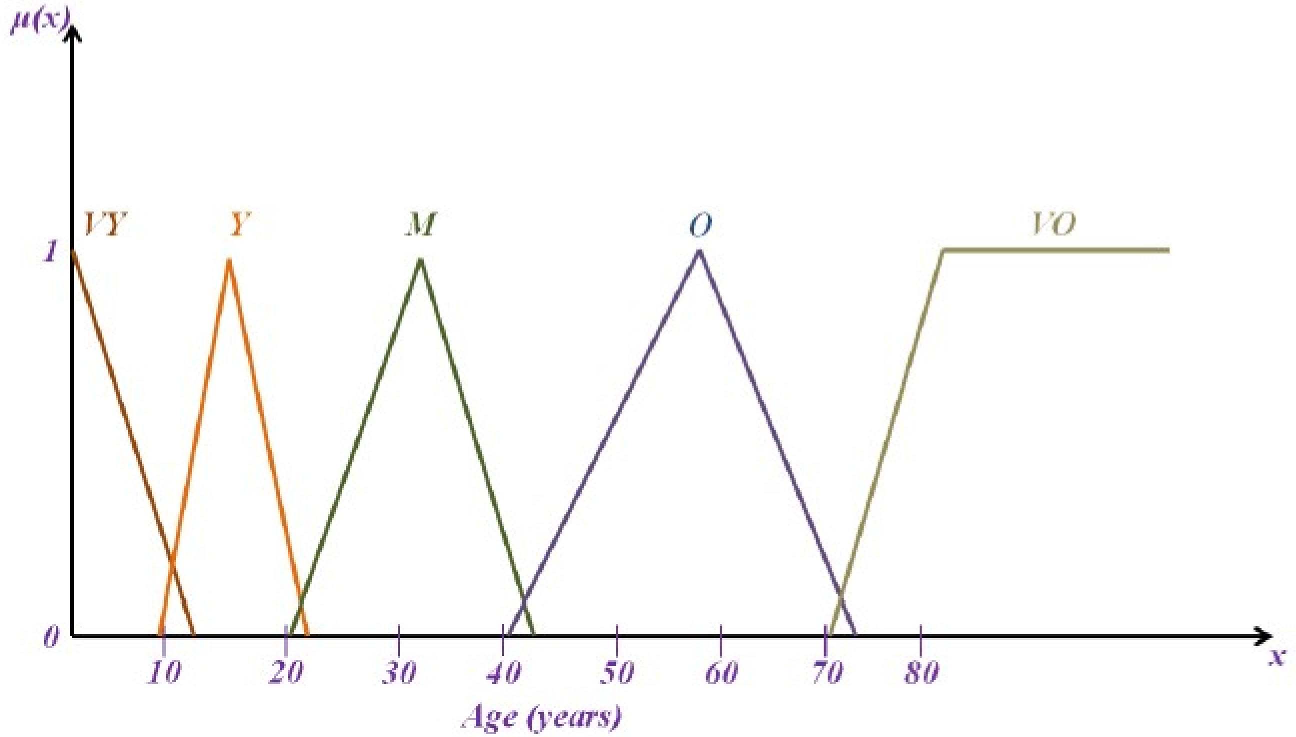
(2) *Using your own intuition and definitions of the uni- verse of discourse, plot fuzzy membership functions for* "*age of people*".

$U = \text{age of people}$

$\text{Let } A \text{ denote age of people in years.}$

$\text{Let the linguistic variables be the following:}$

$$\begin{aligned}
\text{Very young}(VY) &: A < 12 \\
\text{Young}(Y) &: 10 \leq A \leq 22 \\
\text{Middle age}(M) &: 20 \leq A \leq 42 \\
\text{Old}(O) &: 40 \leq A \leq 72 \\
\text{Very old}(VO) &: 70 < A
\end{aligned}$$


μ(x)
VY
Y
M
O
VO
1
0
10
20
30
40
50
60
70
80
x
Age (years)

Inference

- The *inference method* uses knowledge to perform deductive reasoning.
- Deduction achieves conclusion by means of forward inference.
- The knowledge of geometrical shapes and geometry is used for defining membership values.
- The membership functions may be defined using various shapes: *triangular, trapezoidal, bell shaped, Gaussian etc.*
- The inference method here is via triangular

- Consider a triangle, where X, Y and Z are the angles, such that

$$X \geq Y \geq Z \geq 0$$

Let U be the universe of triangles, *ie*,

$$U = \{(X, Y, Z) | X \geq Y \geq Z \geq 0; X + Y + Z = 180\}$$

- There are various types of triangles available:
  1. $I =$ *isosceles triangle*
  2. $E =$ *equilateral triangle*
  3. $R =$ *right-angle triangle*
  4. $IR =$ *isosceles and right-angle triangle*
  5. $T =$ *other triangles*

- The membership values of approximate isosceles triangle is obtained using,

$$\mu_I(X, Y, Z) = 1 - \frac{1}{60^o} min(X - Y, Y - Z)$$

where,

$$X \geq Y \geq Z \geq 0 \text{ and } X + Y + Z = 180$$

If $X = Y$ or $Y = Z$, then

$$\mu_I(X, Y, Z) = 1$$

If $X = 120^o$ or $Y = 60^o$ and $Z = 0^o$, then

$$\mu_I(X, Y, Z) = 0$$

- The membership value of approximate right angle triangle is given by,

$$\mu_R(X, Y, Z) = 1 - \frac{1}{90^o}|X - 90^o|$$

If $X = 90^o$, then

$$\mu_R(X, Y, Z) = 1$$

If $X = 180^o$, then

$$\mu_R(X, Y, Z) = 0$$

- The membership value of approximate isosceles right angle triangle is obtained by,

$$IR = I \cap R$$

and it is given by,

$$\mu_{IR}(X, Y, Z) = min[\mu_I(X, Y, Z), \mu_R(X, Y, Z)]$$
$$= 1 - max[\frac{1}{60^o} min(X - Y, Y - Z), \frac{1}{90^o}|X - 90^o|]$$

- The membership function for a fuzzy equilateral triangle is given by,

$$\mu_E(X, Y, Z) = 1 - \frac{1}{180^o}|X - Z|$$

- The membership function of other triangles is given by,

$$T = \overline{I \cup R \cup E}$$

By using DeMorgan's law,

$$T = \overline{I} \cap \overline{R} \cap \overline{E}$$

The membership value can be obtained using,

$$\mu_T(X, Y, Z) =$$
$$min[1 - \mu_I(X, Y, Z), 1 - \mu_E(X, Y, Z), 1 - \mu_R(X, Y, Z)]$$

Problem

(1) *Using the inference approach, find the membership values for the triangular shapes I,R,E,IR and T for a triangle with angles* $45^o, 55^o$ *and* 180^o.

Let the universe of discourse be,

$$U = \{(X, Y, Z) | X = 80^o \geq Y = 55^o \geq Z = 45^o \geq 0, X + Y + Z = 80^o + 55^o + 45^o = 180^o\}$$

Membership value of isosceles triangle, I

$$\mu_I = 1 - \frac{1}{60^o} min(X - Y, Y - Z)$$
$$= 1 - \frac{1}{60^o} min(80^o - 55^o, 55^o - 45^o)$$
$$= 1 - \frac{1}{60^o} min(25^o, 10^o) = 1 - \frac{1}{60^o} \times 10^o = 0.833$$

Membership value of right angle triangle, R

$$\mu_R = 1 - \frac{1}{90^o} |X - 90^o|$$
$$= 1 - \frac{1}{90^o} |80^o - 90^o|$$
$$= 1 - \frac{1}{90^o} \times 10^o = 0.889$$

Membership value of equilateral triangle, E

$$\mu_E = 1 - \frac{1}{180^o}(X - Z)$$
$$= 1 - \frac{1}{180^o}(80^o - 45^o)$$
$$= 1 - \frac{1}{180^o} \times 35^o = 0.8056$$

Membership value of isosceles and right angle triangle, IR

$$\mu_{IR} = min(\mu_I, \mu_R)$$
$$= min(0.833, 0.889) = 0.833$$

Membership value of other triangles, T

$$\mu_T = min(1 - \mu_I, 1 - \mu_E, 1 - \mu_R)$$
$$= min(0.167, 0.1944, 0.111) = 0.111$$

Rank Ordering

- On the basis of the preferences made by an individual, a committee, a poll and other opinion methods.
- Pairwise comparisons enables to determine preferences.
- This results in determining the order of the membership.

- Eg
- 1. Formation of Government is based on polling concept
- 2. To identify a best student, ranking may be performed
- 3 To buy a car we can seek several opinion

Angular Fuzzy Sets

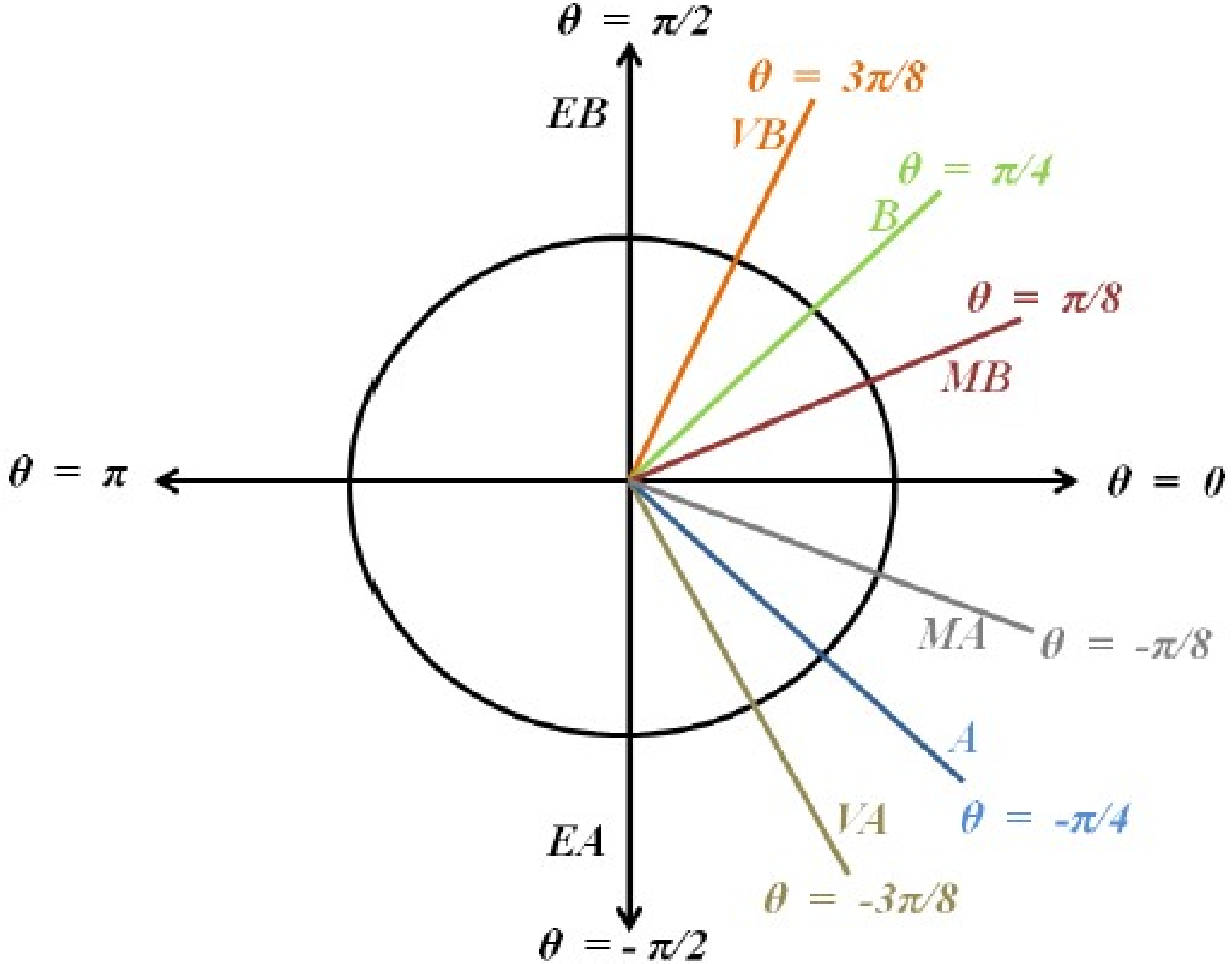
- *Angular fuzzy sets* are defined on a universe of angles, thus repeating the shapes every 2π cycles.
- The truth values of the linguistic variable are represented by angular fuzzy sets.
- The membership value corresponding to the linguistic term can be obtained by,

$$\mu_r(\theta) = t \cdot tan(\theta)$$

where t is horizontal projection of radial vector.ie,

$$t = cos(\theta)$$

Model of angular fuzzy set



- Consider pH value of wastewater from a dyeing industry.
- pH = 7 $\rightarrow$ neutral; below 7 $\rightarrow$ Acid; 7-14 $\rightarrow$ Base
- Linguistic variables are build in such a way that

neutral (N) $\rightarrow$ $\theta = -\Pi/2$ rad

- Base (B)
- Very Base (VB)
- Exact Base(EB)
- Medium Base (MB)
- Acid (A)
- Very Acid (VA)
- Exact Acid(EA)
- Medium Acid (MA)

Neural Networks

- NN can also be used to obtain fuzzy membership values
- Fuzzy MFs are created for input dataset
- I/p data set divided in to training and test data
- Data points are grouped in to clusters
- If a data point belong to a cluster its MF is 1 in that cluster and 0 in other clusters
- NN uses data point marked 1 for training
- When a coordinate location of a point is given, NN assigns a membership value to that point
- NN classifies the point into one of the clusters which is the o/p

Genetic Algorithms

1. For a particular functional mapping system, the same membership functions and shapes are assumed fo various fuzzy variables to be defined.
2. These chosen membership functions are then coded into bit strings.
3. Then these bit strings are concatenated together.
4. The fitness function to be used here is noted. In genetic algorithm, fitness function plays a major rol similar to that played by activation function in neural network.
5. The fitness function is used to evaluate the fitness of each set of membership functions.
6. These membership functions define the functional mapping of the system.

Induction Reasoning

- Uses backward inference
- Employs entropy minimization principle
- Well defined database for i/p – o/p relationship exists
- Can be applied for complex systems where data are abundant and static
- Not suited for dynamic systems
- There are 3 laws of induction

1. Given a set of irreducible outcomes of an experiment, the induced probabilities are those probabilities consistent with all available information that maximize the entropy of the set.
2. The induced probability of a set of independent observations is proportional to the probability density of the induced probability of a single observation.
3. The induced rule is that rule consistent with all available information of that minimizes the entropy.

Membership function is calculated as

1. A fuzzy threshold is to be established between classes of data.
2. Using entropy minimization screening method, first determine the threshold line.
3. Then start the segmentation process.
4. The segmentation process results into two classes.
5. Again partitioning the first two classes one more time, we obtain three different classes.
6. The partitioning is repeated with threshold value calculations, which lead us to partition the data set into a number of classes or fuzzy sets.
7. Then on the basis of the shape, membership function is determined.

Defuzzification

- Defuzzification means the fuzzy to crisp conversion.

- The fuzzy results can not be used in an application, where decision has to be taken only on crisp values.

Example:

If T_{HIGH} then rotate R_{FIRST} .

Here, may be input T_{HIGH} is fuzzy, but action **rotate** should be based on the crisp value of R_{FIRST} .

Defuzzification methods

A number of defuzzification methods are known. Such as

1. **Lambda-cut method**
2. **Weighted average method**
3. **Maxima methods**
4. **Centroid methods**

method

•Lambda-cut method is applicable to derive crisp value.

• It can be applied to both fuzzy set and fuzzy relation

- Lambda-cut method for fuzzy set
- Lambda-cut method for fuzzy relation

Lambda-cut method is alternatively termed as *Alpha-cut method*.

Lamda-cut method for fuzzy set

1. In this method a fuzzy set A is transformed into a crisp set A_λ for a given value of λ $(0 \leq \lambda \leq 1)$

2. In other-words, $A_\lambda = \{x / \mu_A(x) \geq \lambda\}$

3. That is, the value of Lambda-cut set A_λ is x , when the membership value corresponding to x is greater than or equal to the specified λ.

4. This Lambda-cut set A_λ is also called alpha-cut set.

Example

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

Then $A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$

and

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$

Lambda-cut sets : Example

Two fuzzy sets *P* and *Q* are defined on *x* as follows.

$\mu(x)$	x_1	x_2	x_3	x_4	x_5
P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following : (Home Work)

(a) $P_{0.2}$, $Q_{0.3}$

(b) $(P \cup Q)_{0.6}$

(c) $(P \cup \overline{P})_{0.8}$

(d) $(P \cap Q)_{0.4}$

Lamda cut for a Fuzzy Relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of $\lambda = 0, 0.2, 0.9, 0.5$

$$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Properties of λ−cut sets

1. $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$
2. $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
3. $(\overline{R})_\lambda \neq (\overline{R_\lambda})$ *except when* $\lambda = 0.5$
4. *For any* $\lambda \leq \beta$, *where* $0 \leq \beta \leq 1$, *it is true that*, $R_\beta \subseteq R_\lambda$.

Properties of λ- cut in fuzzy relation

If R and S are two fuzzy relations, defined with the same fuzzy sets over the same universe of discourses, then

5. $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$
6. $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
7. $(\overline{R})_\lambda \neq \overline{R}_\lambda$
8. For $\lambda \leq \beta$, where β between 0 and 1 , then $R_\beta \subseteq R_\lambda$

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into crisp set (or relation).

Problems

(1) *Consider two fuzzy sets A and B, both defined on X,given as follows:*

$\mu(x_i X)$	x_1	x_2	x_3	x_4	x_5
A	0.2	0.3	0.4	0.7	0.1
B	0.4	0.5	0.6	0.8	0.9

Express the following $\lambda-$ sets using Zadeh's notation:

$(a)(\overline{A})_{0.7}$ $(b)(B)_{0.2}$

$(c)(A \cup B)_{0.6}$ $(d)(A \cap B)_{0.5}$

$(e)(A \cup \overline{A})_{0.7}$ $(f)(B \cap \overline{B})_{0.3}$

$(g)(\overline{A \cap B})_{0.6}$ $(h)(\overline{A} \cup \overline{B})_{0.8}$

The two fuzzy sets given are,

$$A = \{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\}$$
$$B = \{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\}$$

$$(a)(\overline{A}) = 1 - \mu_A(x) = \{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\}$$
$$(\overline{A})_{0.7} = \{x_1, x_2, x_5\}$$

$$(b)(B)_{0.2} = \{x_1, x_2, x_3, x_4, x_5\}$$

$$(c)(A \cup B) = max\{\mu_A(x), \mu_B(x)\}$$
$$= \{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}\}$$
$$(A \cup B)_{0.6} = \{x_3, x_4, x_5\}$$

$$(d)(A \cap B) = min\{\mu_A(x), \mu_B(x)\}$$
$$= \{\frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.7}{x_4} + \frac{0.1}{x_5}\}$$
$$(A \cap B)_{0.5} = x_4$$

$$(e)(A \cup \overline{A}) = max\{\mu_A(x), \mu_{\overline{A}}(x)\}$$
$$= \{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5}\}$$
$$(A \cup \overline{A})_{0.7} = \{x_1, x_2, x_4, x_5\}$$

$$(f)(\overline{B}) = 1 - \mu_B(x) = \{\frac{0.6}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5}\}$$
$$(B \cap \overline{B}) = min\{\mu_B(x), \mu_{\overline{B}}(x)\} = \{\frac{0.4}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5}\}$$
$$(B \cap \overline{B})_{0.3} = \{x_1, x_2, x_3\}$$

$$(g)(\overline{A \cap B}) = 1 - \mu_{A \cap B}(x) = \{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\}$$

$$(\overline{A \cap B})_{0.6} = \{x_1, x_2, x_3, x_5\}$$

$$(h)(\overline{A} \cup \overline{B}) = max\{\mu_{\overline{A}}(x), \mu_{\overline{B}}(x)\}$$

$$= \{\frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.3}{x_4} + \frac{0.9}{x_5}\}$$

$$(\overline{A} \cup \overline{B})_{0.8} = \{x_1, x_5\}$$

(2) *Consider the discrete fuzzy set defined on the universe,* $X = \{a, b, c, d, e\}$ *as,*

$$A = \{\frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0}{e}\}$$

Find the $\lambda-$ *cut sets for* $\lambda = 1, 0.9, 0.6, 0.3, 0^{+}$ *and* 0.

$$(a)\lambda = 1, A_1 = \{\frac{1}{a} + \frac{0}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e}\}$$

$$(b)\lambda = 0.9, A_{0.9} = \{\frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e}\}$$

$$(c)\lambda = 0.6, A_{0.6} = \{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d} + \frac{0}{e}\}$$

$$(d)\lambda = 0.3, A_{0.3} = \{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e}\}$$

$$(e)\lambda = 0^+, A_{0^+} = \{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e}\}$$

$$(f)\lambda = 0, A_0 = \{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\}$$

(3) *Determine the crisp* $\lambda-$ *cut relation when* $\lambda = 0.1, 0^{+}, 0.3$ *and* 0.9 *for the following relation* R:

$$R = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0.3 & 0.7 & 0.1 \\ 0.8 & 0.9 & 1.0 \end{bmatrix}$$

$(a)\lambda = 0.1$

$$R_{0.1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$(b)\lambda = 0^{+}$

$$R_{0^+} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$(c)\lambda = 0.3$

$$R_{0.3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$(d)\lambda = 0.9$

$$R_{0.9} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Defuzzification methods

1. **Lambda-cut method**
2. **Weighted average method**
3. **Maxima methods**
4. **Centroid methods**

Defuzzification Methods contd.

- **Maxima Methods**
  1. Height method
  2. First of maxima (FoM)
  3. Last of maxima (LoM)
  4. Mean of maxima(MoM)
- **Centroid methods** ✓
  1. Center of gravity method (CoG)
  2. **Center of sum method (CoS)** ✓
  3. Center of area method (CoA)
- **Weighted average method**

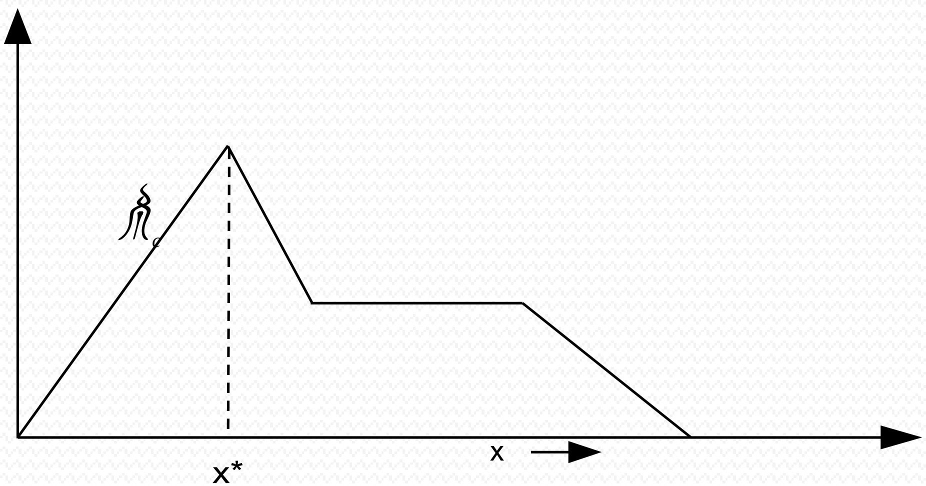
Maxima methods

- Height method
- First of maxima (FoM)
- Last of maxima (LoM)
- Mean of maxima(MoM)

Maxima method : **Height method**

This method is based on Max-membership principle, and defined as follows.

$$\mu_C(x^*) \geq \mu_C(x) \text{ for all } x \in X$$



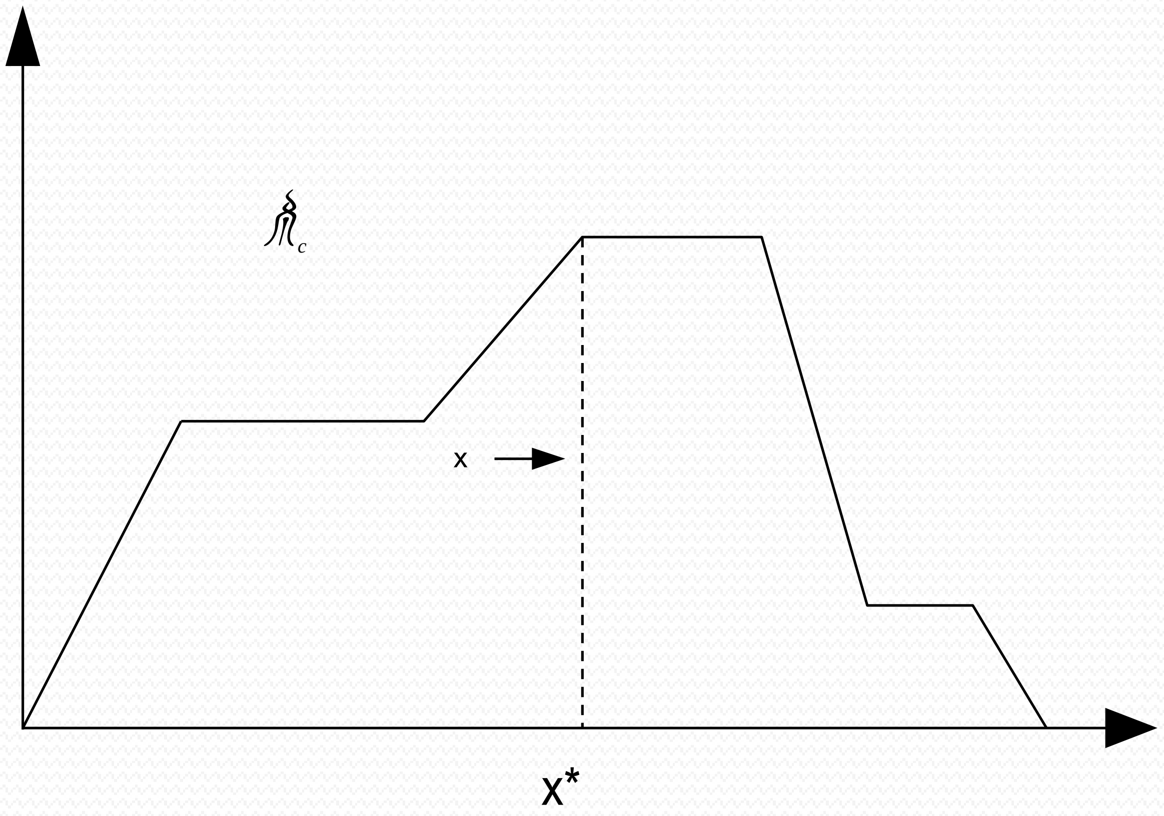
Note:

1. Here, x^* is the height of the output fuzzy set C.

2. This method is applicable when height is unique.

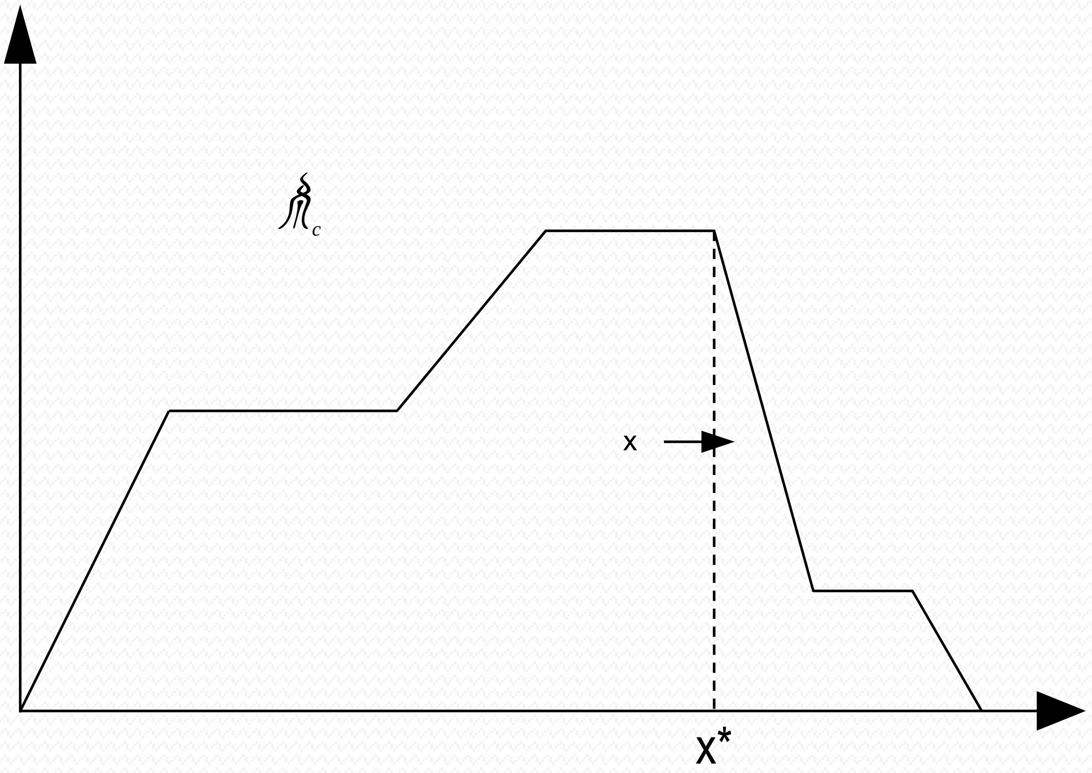
Maxima method : FoM

FoM: First of Maxima : $x^* = min\{x / C(x) = max_w C\{w\}\}$



Maxima method : LoM

LoM : Last of Maxima : $x^* = max\{x / C(x) = max_w C\{w\}\}$



Maxima method : MoM

$$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where, $M = \{x_i | \mu(x_i) = h(C)\}$ where $h(C)$ is the height of the fuzzy set C

MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

Young = {(15,0.5), (20,0.8), (25,0.8), (30,0.5), (35,0.3) }

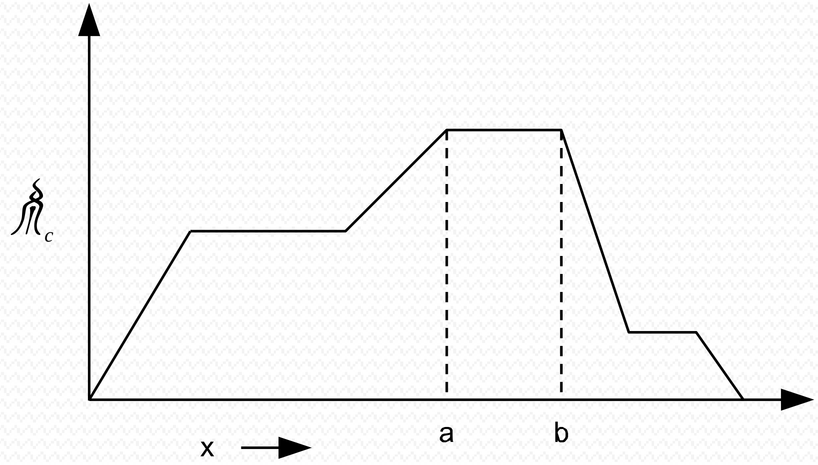
Then the crisp value of **Young** using MoM method is

$$x^* = \frac{20+25}{2} = 22.5$$

Thus, a person of 22.5 years old is treated as young!

MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



$$x^* = \frac{a+b}{2}$$

Note:

- Thus, MoM is also synonymous to middle of maxima.
- MoM is also general method of Height.

Centroid methods

- Center of gravity method (CoG)
- Center of sum method (CoS)
- Center of area method (CoA)

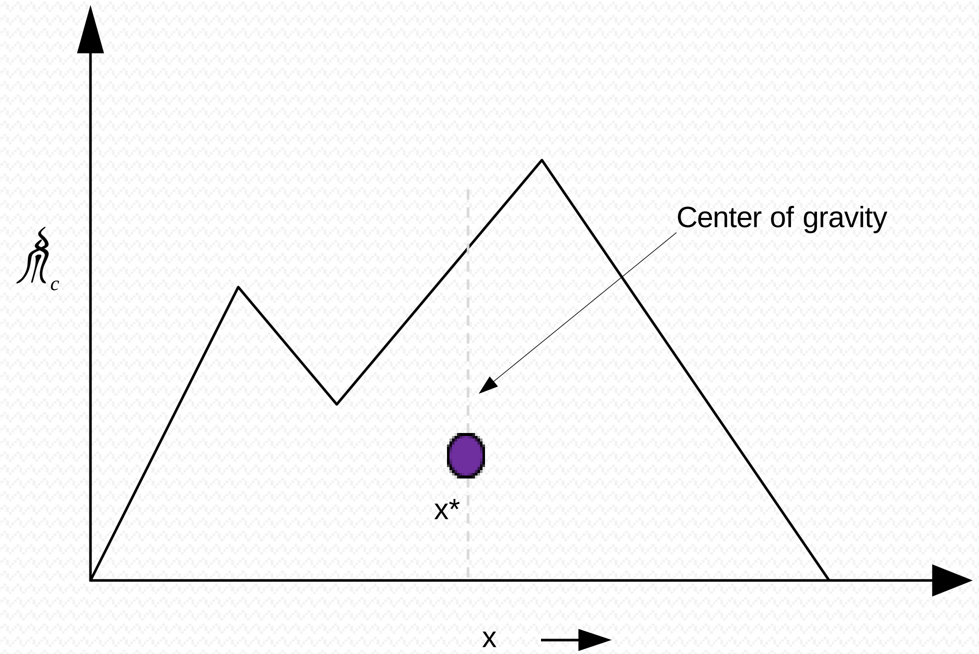
Centroid method : CoG

1. The basic principle in CoG method is to find the point x^* where a vertical line would slice the aggregate into two equal masses.

2. Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$

3. Graphically
,



Centroid method : CoG contd.

1. x^* is the x-coordinate of center of gravity.
2. $\int \mu_C(x)dx$ denotes the area of the region bounded by the curve μ_C.
3. If μ_C is defined with a discrete membership function, then CoG can be stated as :

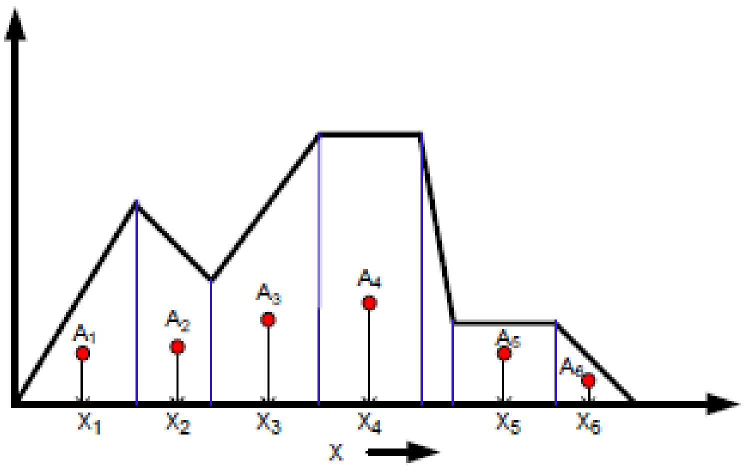
$$x^* = \frac{\sum_{i=1}^{n} x_i.\mu_C(x_i)}{\sum_{i=1}^{n} \mu_C(x_i)} \; ;$$

4. Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

CoG : A geometrical method of calculation

Steps:

1. Divide the entire region into a number of small regular regions (e.g. triangles, trapizoid etc.)

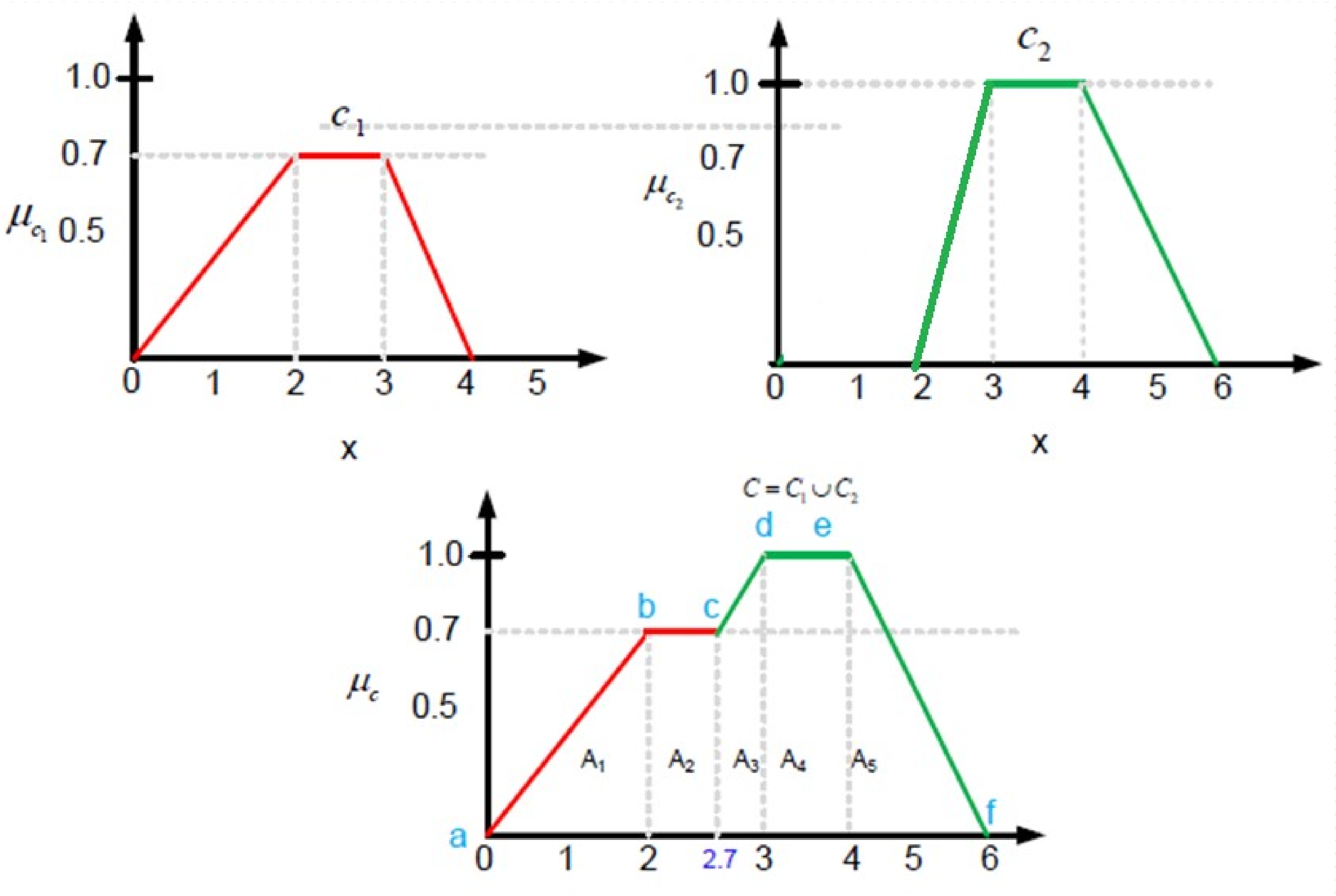


2. Let A_i and x_i denotes the area and c.g. of the i-th portion.
3. Then x^* according to CoG is

$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

where n is the number of smaller geometrical components.

calculation (HW)



CoG: An example of integral method of calcula

$$\mu_c(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x - 2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ (-0.5x + 3) & 4 \le x \le 6 \end{cases}$$

For $A_1 : y - 0 = \frac{0.7}{2}(x - 0)$, or $y = 0.35x$

For $A_2 : y = 0.7$

For $A_3 : y - 0 = \frac{1-0}{3-2}(x - 2)$, or $y = x - 2$

For, $A_4 : y = 1$

For, $A_5 : y - 1 = \frac{0-1}{6-4}(x - 4)$, or $y = -0.5x + 3$

CoG: An example of integral method of calculation (HW)

Thus, $x^* = \frac{\int x.\mu_c(x)dx}{\int \mu_c(x)dx} = \frac{N}{D}$

$N = \int_0^2 0.35x^2 dx + \int_2^{2.7} 0.7x^2 dx + \int_{2.7}^3 (x^2 - 2x)dx + \int_3^4 xdx + \int_4^6 (-0.5x^2 + 3x)dx$

$= 10.98$

$D = \int_0^2 0.35xdx + \int_2^{2.7} 0.7xdx + \int_{2.7}^3 (x - 2)dx + \int_3^4 dx + \int_4^6 (-0.5x + 3)dx$

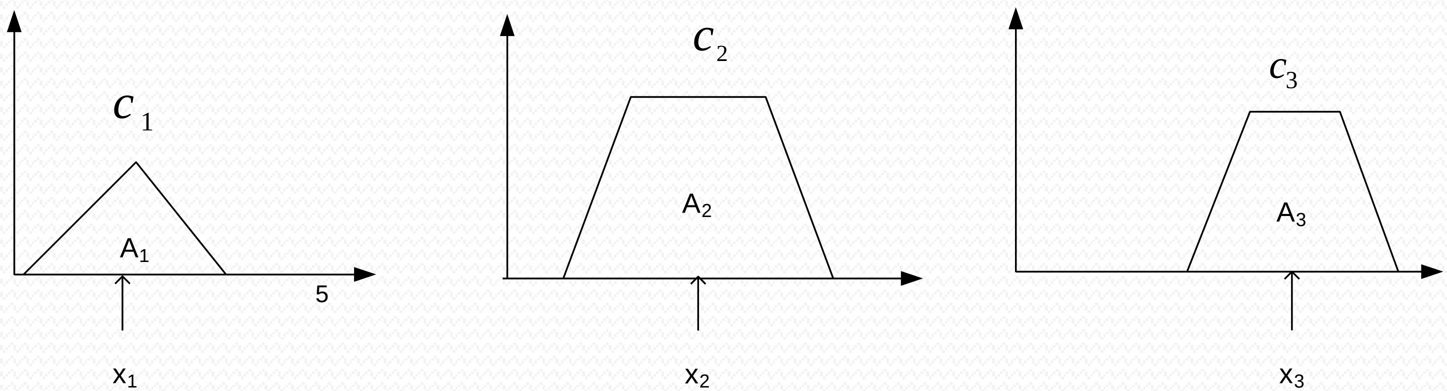
$= 3.445$

Thus, $x^* = \frac{10.98}{3.445} = 3.187$

CoS

If the output fuzzy set $C = C_1 \cup C_2 \cup \ldots C_n$, then the crisp value according to CoS is defined as

$$x^* = \frac{\int_x x \sum_{i=1}^{n} \mu_{C_i}(x)\,dx}{\int_x \sum_{i=1}^{n} \mu_{C_i}(x)\,dx}$$



Centroid method : CoS

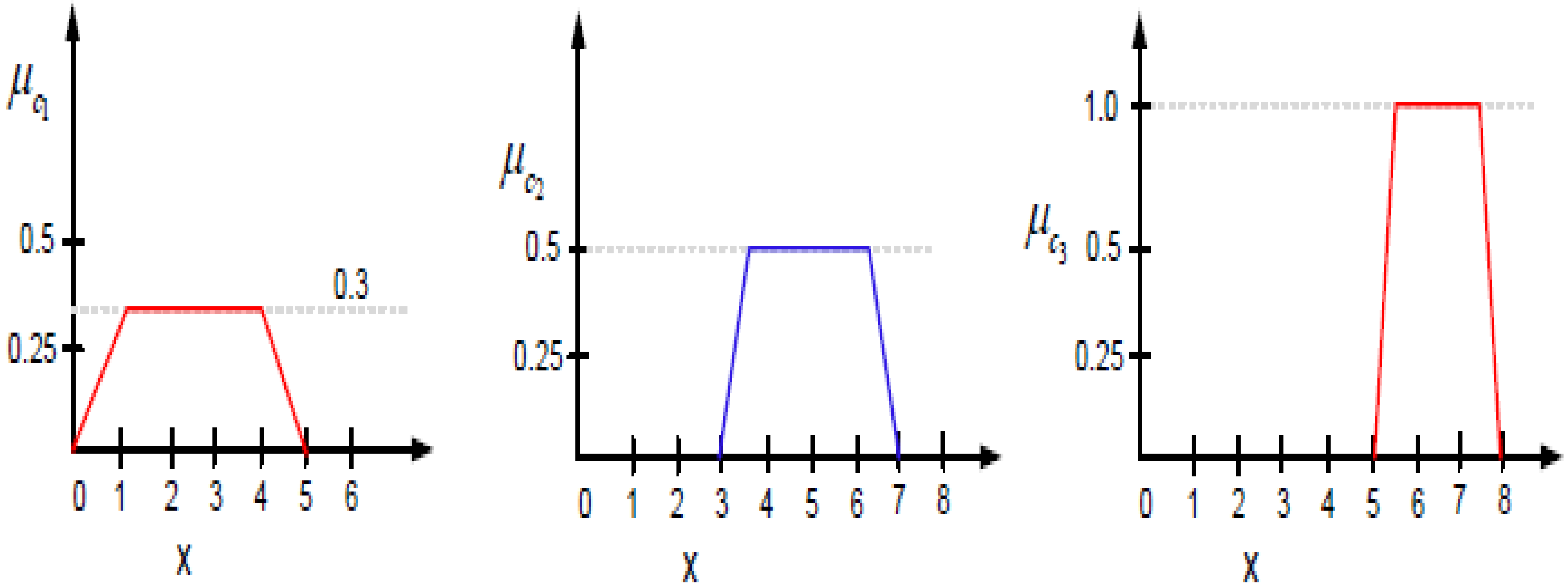
Note:

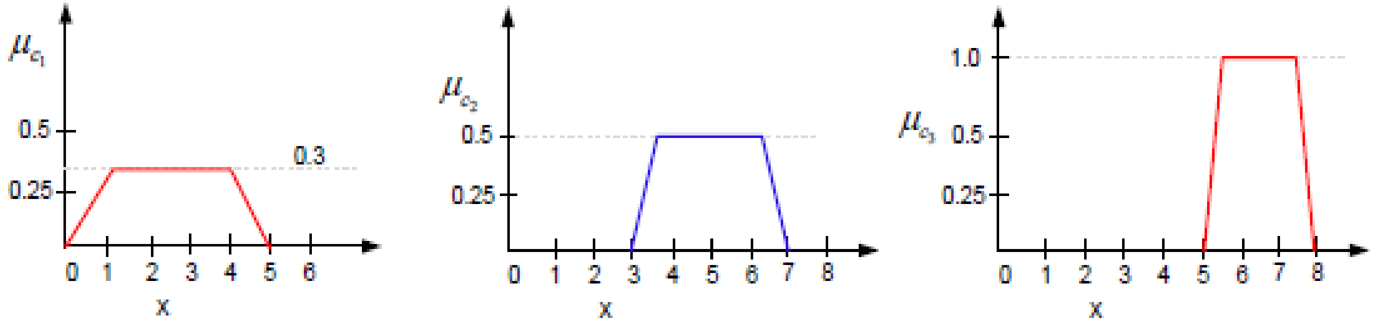
1. In CoG method, the overlapping area is counted once, whereas, in CoS , the overlapping is counted twice or so.

2. In CoS, we use the center of area and hence, its name instead of center of gravity as in CoG.

Example

Consider the three output fuzzy sets as shown in the following plots:





In this case, we have

$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3 + 5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4 + 2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1 \times (3 + 1), x_3 = 6.5$$

Thus, $x^* = \frac{\frac{1}{2}\times0.3\times(3+5)\times2.5+\frac{1}{2}\times0.5\times(4+2)\times5+\frac{1}{2}\times1\times(3+1)\times6.5}{\frac{1}{2}\times0.3\times(3+5)+\frac{1}{2}\times0.5\times(4+2)+\frac{1}{2}\times1\times(3+1)} = 5.00$

Note:

The crisp value of $C = C_1 \cup C_2 \cup C_3$ using CoG method can be found to be calculated as $x^* = 4.9$

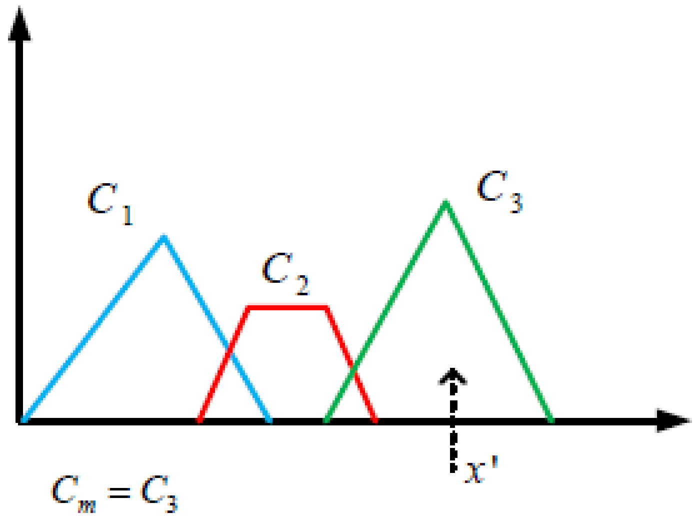
Centroid method: Certer of largest area

If the fuzzy set has two subregions, then the center of gravity of the subregion with the largest area can be used to calculate the defuzzified value.

Mathematically, $x^* = \frac{\int \mu_{c_m}(x).x' dx}{\int \mu_{c_m}(x) dx}$;

Here, C_m is the region with largest area, x' is the center of gravity of C_m.

Graphically,



Weighted average method

1. This method is also alternatively called "Sugeno defuzzification" method.
2. The method can be used only for symmetrical output membership functions.
3. The crisp value accroding to this method is

$$x^* = \frac{\sum_{i=1}^{n} \mu_{C_i}(x_i).(x_i)}{\sum_{i=1}^{n} \mu_{C_i}(x_i)}$$

where, $C_1, C_2, ...C_n$ are the output fuzzy sets and (x_i) is the value where middle of the fuzzy set C_i is observed.

Weighted average method

Graphically

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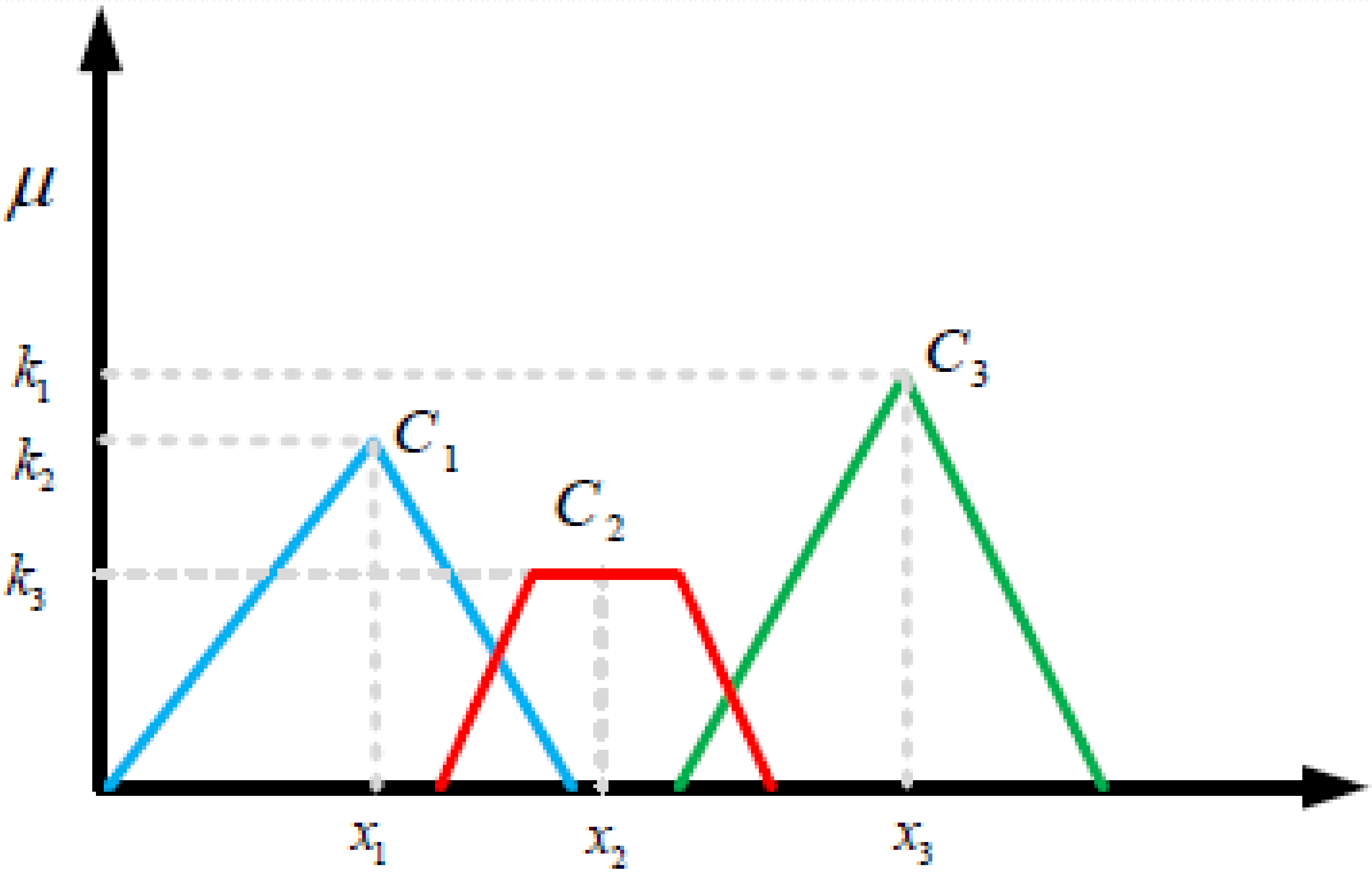

Summary

- CoG $$X^* = \frac{\int x.\mu_C(x)dx}{\int \mu_C(x)dx}$$
- CoA $$X^* = \frac{\int \mu_{c_m}(x).x' dx}{\int \mu_{c_m}(x)dx}$$
-

- *Center of sums method:* The defuzzified value x^* is given by

$$x^* = \frac{\int_x x \sum_{i=1}^{n} \mu_{C_i}(x)\,dx}{\int_x \sum_{i=1}^{n} \mu_{C_i}(x)\,dx}$$

- Weighted Average Method

$$x^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{x}_i) \cdot \bar{x}_i}{\sum \mu_{\underset{\sim}{C}}(\bar{x}_i)}$$